Practical Stabilization of Systems with a Fold Control Bifurcation

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1 Introduction

Nonlinear parameterized dynamical systems exhibit complicated performance around bifurcation points. As the parameter of a system is varied, changes may occur in the qualitative structure of its solutions around an equilibrium point. Usually, this happens when some eigenvalues of the linearized system cross the imaginary axis as the parameter changes [7].

For control systems, a change of some control properties may occur around an equilibrium point, when there is a lack of linear stabilizability at this point. This is called a control bifurcation [21]. A control bifurcation occurs also for unparameterized control systems. In this case, it is the control that plays the parameter's role in parameterized dynamical systems.

The use of feedback to stabilize a system with bifurcation has been studied by several authors, and some fundamental results can be found in [1], [2], [3], [6], [16], [8], [17],[11],[12], [10], the Ph.D. theses [13], [21], [22] and the references therein.

When the uncontrollable modes are on the imaginary axis, asymptotic stabilization of the solution is possible under certain conditions, but when the uncontrollable modes have a positive real part, asymptotic stabilization is impossible to obtain by smooth feedback [4].

In this paper, we show that by combining center manifold techniques with the normal forms approach, it is possible to practically stabilize systems with a fold control bifurcation [21], i.e. those with one slightly unstable uncontrollable mode. The methodology is based on using a class $\mathcal{C}^0$ feedback to obtain a bird foot bifurcation ([20]) in the dynamics of the closed loop system on the center manifold. Systems with a fold control bifurcation appear in applications. For example, in [25] a fold bifurcation appears at the point of passage from minimum phase to nonminimum phase.

W. Kang et al. (Eds.): New Trends in Nonlinear Dynamics and Control, LNCIS 295, pp. 37–48, 2003.

The paper is divided as follows : in Section §1, we introduce definitions of $\varepsilon-$Practical Stability, $\varepsilon-$Practical Stabilizability and Practical Stabilizability ; then, in Section §2, we show that a continuous but non differentiable control law permits the practical stabilization of systems with a fold control bifurcation.

2 Practical Stability and Practical Stabilizability

Practical stability was introduced in [23] and is defined as the convergence of the solution of a differential equation to a neighborhood of the origin. In this section, we propose definitions for practical stability and practical stabilizability.

Let us first define class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions.

Definition 1. *[18, definitions 3.3, 3.4]*

- *A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\lim_{r\to\infty} \alpha(r) = \infty$.*
- *A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r; and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\lim_{r\to\infty} \beta(r, s) = 0$.*

Let $\mathcal{D} \subset \mathbb{R}^n$ be an open set containing the closed ball $\mathbb{B}_\varepsilon$ of radius ε centered at the origin. Let $f : \mathcal{D} \to \mathbb{R}^n$ a continuous function such that $f(0) = 0$.

Consider the system

$$\dot{x} = f(x).$$

Definition 2. *($\varepsilon-$Practical Stability) The origin is said to be locally $\varepsilon-$practically stable, if there exists an open set $\mathcal{D}$ containing the closed ball $\mathbb{B}_\varepsilon$, a class $\mathcal{KL}$ function ζ and a positive constant $\delta = \delta(\varepsilon)$, such that for any initial condition $x(0)$ with $||x(0)|| < \delta$, the solution $x(t)$ of (2) exists and satisfies*

$$d_{\mathbb{B}_\varepsilon}(x(t)) \le \zeta(d_{\mathbb{B}_\varepsilon}(x(0)), t), \quad \forall t \ge 0,$$

with $d_{\mathbb{B}_\varepsilon}(x(t)) = \inf_{\rho\in\mathbb{B}_\varepsilon} d(x(t), \rho)$, the usual point to set distance.

Now consider the controlled system

$$\dot{x} = f(x, v),$$

with $f : \mathcal{D} \times \mathcal{U} \to \mathbb{R}^n$, $f(0, 0) = 0$, and $\mathcal{U} \subset \mathbb{R}^m$ a domain that contains the origin.

Definition 3. *($\varepsilon-$Practical Stabilizability) The system (2) is said to be locally $\varepsilon-$practically stabilizable around the origin, if there exists a control law $v = k_\varepsilon(x)$, such that the origin of the closed-loop system $\dot{x} = f(x, k_\varepsilon(x))$ is locally $\varepsilon-$practically stable.*

Definition 4. *(Practical Stabilizability) The system (2) is said to be locally practically stabilizable around the origin, if it is locally $\varepsilon-$practically stabilizable for every $\varepsilon > 0$.*

If, in the preceding definitions, $\mathcal{D} = \mathbb{R}^n$, then the corresponding properties of $\varepsilon-$practical stability, $\varepsilon-$practical stabilizability and practical stabilizability are global, and the adverb "locally" is omitted.

Now, let us reformulate the local $\varepsilon-$practical stability in the Lyapunov framework.

Let V be a function $V : \mathcal{D} \to \mathbb{R}^+$, such that V is smooth on $\mathcal{D} \setminus \mathbb{B}_\varepsilon$, and satisfies

$$x \in \mathcal{D} \Longrightarrow \alpha_1(d_{\mathbb{B}_\varepsilon}(x(t))) \ \leq V(x) \leq \alpha_2(d_{\mathbb{B}_\varepsilon}(x(t))),$$

with α_1 and α_2 class $\mathcal{K}$ functions.

Such a function is called a Lyapunov function with respect to $\mathbb{B}_\varepsilon$, if there exists a class $\mathcal{K}$ function α_3 such that

$$\dot{V}(x) = L_{f(x)}V(x) \leq -\alpha_3(d_{\mathbb{B}_\varepsilon}(x)), \text{ for } x \in \mathcal{D} \setminus \mathbb{B}_\varepsilon.$$

Proposition 1. *The origin of system (2) is $\varepsilon-$practically stable if and only if there exists a Lyapunov function with respect to $\mathbb{B}_\varepsilon$.*

Proof. In [24], the authors gave stability results for systems with respect to a closed/compact, invariant set $\mathcal{A}$. In particular, the definition of asymptotic stability and Lyapunov function with respect to $\mathcal{A}$ were given. In the case where[1] $\mathcal{A} = \mathbb{B}_\varepsilon$, asymptotic stability with respect to $\mathbb{B}_\varepsilon$ reduces to our definition of $\varepsilon-$practical stability (definition 2). The proof of our proposition is obtained by applying a local version of [24, Theorem 2.9]. ◇

If $\mathcal{D} = \mathbb{R}^n$ and α_1, α_2 are a class $\mathcal{K}_\infty$ functions, the origin is globally $\varepsilon-$practically stable.

Remark : When $\varepsilon = 0$, we recover the classical definitions of local and global asymptotic stability.

3 Systems with a Fold Control Bifurcation

In this section, we apply the ideas of the preceding section to the system (2) when its linearization is uncontrollable at an equilibrium, which we take to

[1] $\mathbb{B}_\varepsilon$ is a nonempty, compact and invariant set. It is invariant since $\dot{V}$ is negative on its boundary ; so, a solution starting in $\mathbb{B}_\varepsilon$ remains in it.

be the origin. Suppose that $m = 1$ and that the linearization of the system (2) at the origin is (A, B) with

$$A = \frac{\partial f}{\partial x}(0,0), \quad B = \frac{\partial f}{\partial v}(0,0),$$

and

$$\mathrm{rank}([B\ AB\ A^2B\ \cdots\ A^{n-1}B]) = n-1.$$

According to the assumption in (3), the linear system is uncontrollable. Suppose that the uncontrollable mode λ satisfies the following assumption
Assumption : The uncontrollable mode is $\lambda \in \mathbb{R}_{\geq 0}$.

Let us denote as $\Sigma_{\mathcal{U}}$, the system (2) under the above assumption. This system exhibits a fold control bifurcation when $\lambda > 0$, and, generically, a transcontrollable bifurcation when $\lambda = 0$ (see [21]).

From linear control theory [14], we know that there exist a linear change of coordinates and a linear feedback that put the system $\Sigma_{\mathcal{U}}$ in the following form

$$\begin{aligned}\dot{\bar{z}}_1 &= \lambda \bar{z}_1 + O(\bar{z}_1, \bar{z}_2, \bar{u})^2,\\ \dot{\bar{z}}_2 &= A_2\bar{z}_2 + B_2\bar{u} + O(\bar{z}_1, \bar{z}_2, \bar{u})^2,\end{aligned}$$

with $\bar{z}_1 \in \mathbb{R}$, $\bar{z}_2 \in \mathbb{R}^{(n-1)\times 1}$, $A_2 \in \mathbb{R}^{(n-1)\times(n-1)}$ and $B_2 \in \mathbb{R}^{(n-1)\times 1}$. The matrices A_2 and B_2 are given by

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix}, \quad B_2 = \begin{bmatrix}0\\0\\ \vdots\\0\\1\end{bmatrix}.$$

To simplify the quadratic part, we use the following quadratic transformations in order to transform the system to its quadratic normal form

$$\begin{bmatrix} z_1\\ z_2\end{bmatrix} = \begin{bmatrix}\bar{z}_1\\ \bar{z}_2\end{bmatrix} - \phi^{[2]}(\bar{z}_1, \bar{z}_2), \tag{1}$$

$$u = \bar{u} - \alpha^{[2]}(\bar{z}_1, \bar{z}_2, \bar{u}). \tag{2}$$

The normal form is given in the following theorem

Theorem 1. *[21, Theorem 2.1] For the system $\Sigma_{\mathcal{U}}$ whose linear part is of the form (3), there exist a quadratic change of coordinates (1) and feedback (2) which transform the system to*

$$\dot{z}_1 = \lambda z_1 + \beta z_1^2 + \gamma z_1 z_{2,1} + \sum_{j=1}^{n} \delta_j\, z_{2,j}^2 + O(z_1, z_2, u)^3,$$
$$\dot{z}_2 = A_2 z_2 + B_2 u + \sum_{i=1}^{n-1} \sum_{j=i+2}^{n} \theta_i^j\, z_{2,j}^2\, e_2^i + O(z_1, z_2, u)^3,$$

with $\beta, \gamma, \delta_j, \theta_i^j$ are constant coefficients, $z_{2,n} = u$, and e_2^i is the $i^{th}-$ unit vector in the z_2-space.

Let us consider the piecewise linear feedback

$$u = K_1(z_1) z_1 + K_2 z_2 + O(z_1, z_2)^2,$$

with

$$K_1(z_1) = \begin{cases} \bar{k}_1, & z_1 \geq 0, \\ \tilde{k}_1, & z_1 < 0. \end{cases}$$

We wish to stabilize the system around the bifurcation point. The controllable part can be made asymptotically stable by choosing K_2 such that
Property $\mathcal{P}$: The matrix $\bar{A}_2 = A_2 + B_2 K_2$ is Hurwitz.

Under the feedback (3), the system $\Sigma_{\mathcal{U}}$ has $n-1$ eigenvalues with negative real parts, and one eigenvalue with positive real part, the uncontrollable mode λ. Nevertheless, if we view the system $\Sigma_{\mathcal{U}}$ as being parameterized by λ, and by considering λ as an extra-state, satisfying the equation $\dot{\lambda} = 0$, the system $\Sigma_{\mathcal{U}}$ under the feedback (3) possesses two eigenvalues with zero real part and $n-1$ eigenvalues in the left half plane.

Theorem 2. *Consider the closed-loop system (1)-(3), then there exists a center manifold defined by $z_2 = \Pi(z_1, \lambda)$ whose linear part is determined by the feedback (3).*

Proof. By considering λ as an extra state, the linear part of the dynamics (1)-(3) is given by

$$\begin{aligned} \dot{\lambda} &= 0, \\ \dot{z}_1 &= O(\lambda, z_1, z_2)^2, \\ \dot{z}_2 &= B_2 K_1(z_1) z_1 + \bar{A}_2 z_2 + O(z_1, z_2)^2. \end{aligned}$$

λz_1 is now considered as a second order term.

Let $\Sigma_{\bar{k}_1}$ (resp. $\Sigma_{\tilde{k}_1}$) be the system (3) when $K_1(z_1) = \bar{k}_1$ (resp. $K_1(z_1) = \tilde{k}_1$) for all z_1. Since the system $\Sigma_{\bar{k}_1}$ (resp. $\Sigma_{\tilde{k}_1}$) is smooth, and possesses two eigenvalues on the imaginary axis and $n-2$ eigenvalues in the open left half plane ; then, from the center manifold theorem, in a neighborhood of the origin, $\Sigma_{\bar{k}_1}$ (resp. $\Sigma_{\tilde{k}_1}$) has a center manifold $\overline{W}^c$ (resp. $\widetilde{W}^c$).

For $\Sigma_{\bar{k}_1}$, the center manifold is represented by $z_2 = \overline{\Pi}(\lambda, z_1)$, for λ and z_1 sufficiently small. Its equation is

$$\begin{aligned}\dot{z}_2 &= A_2\overline{\Pi}(\lambda, z_1) + B_2(\bar{k}_1 z_1 + K_2\overline{\Pi}(\lambda, z_1)) + O(z_1, z_2)^2,\\ &= \frac{\partial\,\overline{\Pi}(\lambda, z_1)}{\partial\, z_1}\,\dot{z}_1 = O(z_1, z_2)^2.\end{aligned}$$

Since $\dot{\lambda} = 0$ and λz_1 is a second order term in the enlarged space (λ, z_1, z_2), then there is no linear term in λ in the linear part of the center manifold. Hence, the linear part of the center manifold is of the form $z_2 = \overline{\Pi}^{[1]} z_1$, and its $i-$th component is $z_{2,i} = \overline{\Pi}_i^{[1]} z_1$, for $i = 1, \cdots, n-1$. Using (3) we obtain that $\overline{\Pi}_1^{[1]} = -\frac{\bar{k}_1}{k_{2,1}}$ and $\overline{\Pi}_i^{[1]} = 0$, for $2 \le i \le n-1$.

Similarly for $\Sigma_{\tilde{k}_1}$, the center manifold is represented by $z_2 = \widetilde{\Pi}(\lambda, z_1)$. Its linear part is given by $z_2 = \widetilde{\Pi}^{[1]} z_1$, whose components are defined by $\widetilde{\Pi}_1^{[1]} = -\frac{\tilde{k}_1}{k_{2,1}}$ and $\widetilde{\Pi}_i^{[1]} = 0$, for $2 \le i \le n-1$.

Since $\bar{A}_2$ has no eigenvalues on the imaginary axis, and $k_{2,1}$ is the product of all the eigenvalues of $\bar{A}_2$, then $k_{2,1} \neq 0$.

The center manifolds $\overline{W}^c$ and $\widetilde{W}^c$ intersect along the line $z_1 = 0$. Indeed, since $\dot{\lambda} = 0$, then $\frac{\partial^k \overline{\Pi}(\lambda, z_1)}{\partial\,\lambda^k}|_{\lambda=0, z_1=0} = 0$ and $\frac{\partial^k \widetilde{\Pi}(\lambda, z_1)}{\partial\,\lambda^k}|_{\lambda=0, z_1=0} = 0$, for $k \ge 1$. So, $\overline{\Pi}(\lambda, z_1)|_{z_1=0} = 0$ and $\widetilde{\Pi}(\lambda, z_1)|_{z_1=0} = 0$, for all λ.

Hence, if we slice them along the line $z_1 = 0$ and then glue the part of $\overline{W}^c$ for which $z_1 > 0$ with the part of $\widetilde{W}^c$ for which $z_1 < 0$, along this line, we deduce that in an open neighborhood of the origin, $\overline{\mathcal{D}}$, the piecewise smooth system (3) has a piecewise smooth center manifold W_c. The linear part of the center manifold W_c is represented by $z_2 = \Pi^{[1]} z_1$. The $i-th$ component of z_2, $z_{2,i}$, is given by

$$z_{2,i} = \Pi_i^{[1]}(z_1) z_1,$$

with

$$\Pi_1^{[1]}(z_1) = -\frac{K_1(z_1)}{k_{2,1}} \text{ and } \Pi_i^{[1]}(z_1) = 0, \text{ for } i \ge 2.$$

◇

Using (1) and (3), the reduced dynamics on the center manifold is given by

$$\dot{z}_1 = \begin{cases} \lambda z_1 + \Phi(\overline{\Pi}_1^{[1]}) z_1^2 + O(z_1^3),\ z_1 \ge 0,\\ \lambda z_1 + \Phi(\widetilde{\Pi}_1^{[1]}) z_1^2 + O(z_1^3),\ z_1 < 0,\end{cases}$$

with Φ the function defined by $\Phi(X) = \beta + \gamma X + \delta_1 X^2$.

The following theorem shows that the origin of the system (3) can be made practically stable, for small $\lambda > 0$, and asymptotically stable if $\lambda = 0$.

Theorem 3. *Consider system (1) with $\gamma^2 - 4\beta\delta_1 > 0$, then, the piecewise linear feedback (3) practically stabilizes the system around the origin for small $\lambda > 0$, and locally asymptotically stabilizes the system when $\lambda = 0$.*

Proof. See appendix. ◇

If we choose $\overline{\Pi}_1^{[1]}$ and $\widetilde{\Pi}_1^{[1]}$ such that $\Phi(\overline{\Pi}_1^{[1]}) = -\Phi(\widetilde{\Pi}_1^{[1]}) = \Phi_0$, the dynamics (3) will be of the form

$$\dot{z}_1 = \mu z_1 - \Phi_0 |z_1| z_1 + O(z_1^3), \tag{3}$$

with $\mu \in \mathbb{R}$ a parameter. The equation (3) is the normal form of the bird foot bifurcation, introduced by Krener in [20].

If $\Phi_0 > 0$, the equation (3) corresponds to a *supercritical bird foot bifurcation.* For $\mu < 0$, there is one equilibrium at $z_1 = 0$ which is exponentially stable. For $\mu > 0$, there are two exponentially stable equilibria at $z_1 = \pm\frac{\mu}{\Phi_0}$, and one exponentially unstable equilibrium at $z_1 = 0$. For $\mu = 0$, there is one equilibrium at $z_1 = 0$ which is asymptotically stable but not exponentially stable.

If $\Phi_0 < 0$, the equation (3) is an example of *subcritical bird foot bifurcation.* For $\mu < 0$, there is one equilibrium at $z_1 = 0$ which is exponentially stable and two exponentially unstable equilibria at $z_1 = \pm\frac{\mu}{\Phi_0}$. For $\mu > 0$, there is one exponentially unstable equilibrium at $z_1 = 0$. For $\mu = 0$, there is one equilibrium at $z_1 = 0$ which is unstable.

Notice that both normal forms are invariant under the transformation $z_1 \to -z_1$ and so the bifurcation diagrams can be obtained by reflecting the upper or lower half of the bifurcation diagram of a transcritical bifurcation. In both cases the bifurcation diagrams look like the foot of a bird.

In the $\lambda - z_1$ plane, the dynamics (3) are in the form (3) with $\Phi_0 > 0$. A supercritical birdfoot bifurcation appears at $(\lambda, z_1) = (0, 0)$. For $\lambda > 0$, we have 3 equilibrium points : the origin and $\pm\varepsilon$ (corresponding to the solutions of $\dot{z}_1 = 0$). The origin is unstable for $\lambda > 0$, and the two other equilibrium points are stable (cf. Figure 1). The practical stabilization of the system is made possible by making the two new equilibrium points sufficiently close to the origin, i.e. by choosing $\Phi(\overline{\Pi}_1^{[1]})$ and $\Phi(\widetilde{\Pi}_1^{[1]})$ sufficiently large.

If a quadratic feedback was used instead of (3), i.e.

$$u = \bar{K}_1 z_1 + K_2 z_2 + z_1^T Q_{fb} z_1 + O(z_1^3),$$

we can prove that the closed loop dynamics has a center manifold. Moreover, by appropriately choosing $\bar{K}_1$, the reduced dynamics on the center manifold will have the form

$$\dot{z}_1 = \lambda z_1 - \Phi_1 z_1^3 + O(z_1^4),$$

with $\Phi_1 > 0$, by appropriately choosing Q_{fb}.

The equation (3) is the normal form of a system exhibiting a supercritical pitchfork bifurcation. By using a similar analysis as above, we deduce that the solution of the reduced dynamics converges to the equilibrium points $\varepsilon = \pm\sqrt{\frac{\lambda}{\Phi_1}}$, and that the closed-loop system (1)-(3) is practically stabilizable.

The reason of the choice of a piecewise linear feedback instead of a quadratic feedback is that it is preferable to have a supercritical bird foot bifurcation than a supercritical pitchfork bifurcation. This is due to the fact that the stable equilibria in a system with a bird foot bifurcation grow like μ not like $\sqrt{\mu}$ as in the pitchfork bifurcation[2], and that the bird foot bifurcation is robust to small quadratic perturbations, while these transform the pitchfork bifurcation to a transcritical one.

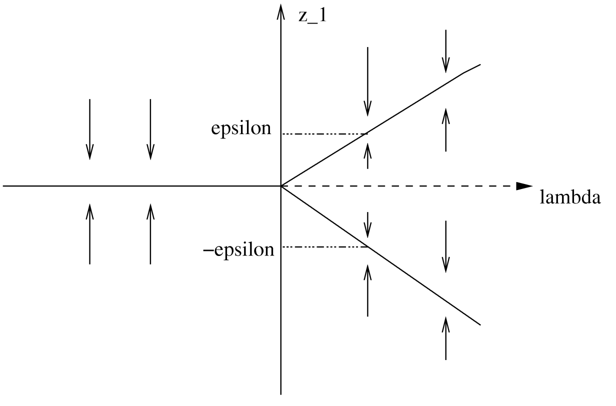

Fig. 1.

4 Appendix

Proof of Theorem 3

Consider the Lyapunov function $V(z_1) = \frac{1}{2}z_1^2$, and let $\varepsilon_1 \triangleq -\frac{\lambda}{\Phi(\overline{\Pi}_1^{[1]})}$ and $\varepsilon_2 \triangleq -\frac{\lambda}{\Phi(\widetilde{\Pi}_1^{[1]})}$. Then, from (3), we have

$$\dot{V} = \begin{cases} \Phi(\overline{\Pi}_1^{[1]})(z_1 - \varepsilon_1)z_1^2 + O(z_1^4),\ z_1 \geq 0, \\ \Phi(\widetilde{\Pi}_1^{[1]})(z_1 - \varepsilon_2)z_1^2 + O(z_1^4),\ z_1 < 0, \end{cases}$$

• Practical Stabilization for $\lambda > 0$

By choosing[3] $\overline{\Pi}_1^{[1]}$ and $\widetilde{\Pi}_1^{[1]}$ such that $\Phi(\overline{\Pi}_1^{[1]}) < 0$ and $\Phi(\widetilde{\Pi}_1^{[1]}) > 0$, we get $\varepsilon_1 > 0$ and $\varepsilon_2 < 0$. This choice is always possible since Φ is a second order polynomial whose discriminant, $\gamma^2 - 4\beta\delta_1$, is positive ; so, Φ takes both positive and negative values. In this case, $\dot{V} < 0$ for $z_1 > \varepsilon_1$ and $z_1 < \varepsilon_2$, and $\dot{V} = 0$ for $z_1 = \varepsilon_1$ or $z_1 = \varepsilon_2$.

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[2] Let us recall that the normal for of a pitchfork bifurcation is $\dot{z}_1 = \mu z_1 - \Phi_1 z_1^3$, with $\mu \in \mathbb{R}$ the parameter.

[3] This choice is made by fixing the parameters $\bar{k}_1$ and $\tilde{k}_1$ of the feedback (3) linked to $\overline{\Pi}_1^{[1]}$ and $\widetilde{\Pi}_1^{[1]}$ through (3).

In the following, and without loss of generality, we choose $\overline{\Pi}_1^{[1]}$ and $\widetilde{\Pi}_1^{[1]}$ such that $\Phi(\overline{\Pi}_1^{[1]}) = -\Phi(\widetilde{\Pi}_1^{[1]})$, so $\varepsilon_1 = -\varepsilon_2 \triangleq \varepsilon$, with $0 \leq \varepsilon \leq r$, and r is the radius of $\mathbb{B}_r$, the largest closed ball contained in $\overline{\mathcal{D}}$.

Let Ω_1 and Ω_2 be two sets defined by $\Omega_1 =]\varepsilon, +r]$ and $\Omega_2 = [-r, -\varepsilon[$.

If $z_1(0) \in \Omega_1 \cup \Omega_2$, and since $\dot{V} < 0$ on $\Omega_1 \cup \Omega_2$, then, from (2) and (2), $\dot{V}$ satisfies

$$\dot{V} \leq -\alpha_3(||z_1||) \leq -\alpha_3(\alpha_2^{-1}(V)).$$

Since α_2 and α_3 are a class $\mathcal{K}$ functions, then $\alpha_3(\alpha_2^{-1})$ is also a class $\mathcal{K}$ function. Hence, using the comparison principle in [24, lemma 4.4], there exists a class $\mathcal{KL}$ function η such that

$$V(z_1(t)) \leq \eta(V(z_1(0), t)).$$

The sets $\overline{\Omega}_1 = [0, \varepsilon]$ and $\overline{\Omega}_2 = [-\varepsilon, 0]$ have the property that when a solution enters either set, it remains in it. This is due to the fact that $\dot{V}$ is negative definite on the boundary of these two sets. For the same reason, if $z_1(0) \in \overline{\Omega}_1$ (resp. $z_1(0) \in \overline{\Omega}_2$), then $z_1(t) \in \overline{\Omega}_1$ (resp. $z_1(t) \in \overline{\Omega}_2$), for $t \geq 0$.

Let $T(\varepsilon)$ be the first time such that the solution enters $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \mathbb{B}_\varepsilon$. Using (2) and (4), we get that for $0 \leq t \leq T(\varepsilon)$,

$$\varepsilon \leq ||z_1(t)|| \leq \alpha_1^{-1}(V(z_1(t)) \leq \alpha_1^{-1}(\eta(V(z_1(0), t))) \triangleq \zeta(z_1(0), t).$$

The function ζ is a class $\mathcal{KL}$ function, since α_1 is a class $\mathcal{K}$ function and η a class $\mathcal{KL}$ function. Since ζ is a class $\mathcal{KL}$ function, then $T(\varepsilon)$ is finite. Hence, $z_1(t) \in \overline{\Omega}_1 \cup \overline{\Omega}_2$, for $t \geq T(\varepsilon)$.

Hence, for $z_1 \in \mathbb{B}_r$, the solution satisfies

$$d_{\mathbb{B}_\varepsilon}(z_1(t)) \leq \zeta(d_{\mathbb{B}_\varepsilon}(z_1(0)), t).$$

So, in $\mathbb{B}_r$, the origin is locally $\varepsilon-$practically stable.

Now, consider the whole closed-loop dynamics

$$\begin{aligned}
\dot{z}_1 &= \lambda z_1 + \beta z_1^2 + \gamma z_1 z_{2,1} + \sum_{i=1}^{n-1} \delta_i \, z_{2,i}^2 + O(z_1, z_2)^3, \\
\dot{z}_2 &= B_2 K_1 z_1 + \bar{A}_2 z_2 + \sum_{i=1}^{n-1} \sum_{j=i+2}^{n-1} \theta_i^j \, z_{2,j}^2 \, e_2^i + O(z_1, z_2)^3.
\end{aligned}$$

Let $w_1 = z_1$, $w_2 = z_2 - \Pi^{[1]} z_1$, and $w = (w_1, w_2)^T$. Then, the closed-loop dynamics is given by

$$\dot{w}_1 = \begin{cases} \lambda w_1 + \Phi(\overline{\Pi}_1^{[1]}) w_1^2 + \bar{\mathcal{N}}_1(w_1, w_2), \text{ for } w_1 \geq 0, \\ \lambda w_1 + \Phi(\tilde{\Pi}_1^{[1]}) w_1^2 + \tilde{\mathcal{N}}_1(w_1, w_2), \text{ for } w_1 < 0. \end{cases}$$

$$\dot{w}_2 = \begin{cases} \bar{A}_2 w_2 + \bar{\mathcal{N}}_2(w_1, w_2), \text{ for } w_1 \geq 0, \\ \bar{A}_2 w_2 + \tilde{\mathcal{N}}_2(w_1, w_2), \text{ for } w_1 < 0. \end{cases}$$

Let

$$\mathcal{N}_i(w_1, w_2) = \begin{cases} \bar{\mathcal{N}}_i(w_1, w_2), & w_1 \geq 0, \\ \tilde{\mathcal{N}}_i(w_1, w_2), & w_1 < 0, \end{cases} \quad \text{for } \ i = 1, 2,$$

with $\mathcal{N}_1(w_1, w_2) = (\gamma + 2\delta_1 \Pi_1^{[1]}) w_1 w_{2,1} + \sum_{i=1}^{n-1} \delta_i \, w_{2,i}^2$ and $\mathcal{N}_2(w_1, w_2) = \sum_{i=1}^{n-1} \sum_{j=i+2}^{n-1} \theta_i^j \, w_{2,j}^2 \, e_2^i$.

Since $\mathcal{N}_i(w_1, 0) = 0$ and $\frac{\partial \mathcal{N}_i}{\partial w_2}(0, 0) = 0$ $(i = 1, 2)$, then in the domain $||w||_2 < \sigma$, $\mathcal{N}_1$ and $\mathcal{N}_2$ satisfy

$$\mathcal{N}_i(w_1, w_2) \leq \kappa_i ||w_2||, \quad i = 1, 2,$$

where κ_1 and κ_2 can be arbitrarily small by making σ sufficiently small.

Since $\bar{A}_2$ is Hurwitz, there exists a unique P such that $A_2^T P + P A_2 = -I$. Let $\mathcal{V}$ be the following composite Lyapunov function

$$\mathcal{V}(w_1, w_2) = \frac{1}{2} w_1^2 + w_2^T P w_2.$$

The derivative of $\mathcal{V}$ along the trajectories of the system is given by

$$\dot{\mathcal{V}}(w_1, w_2) = \pi(w_1) + w_1 \mathcal{N}_1(w_1, w_2) + w_2^T (\bar{A}_2^T P + P \bar{A}_2) w_2 + 2 w_2^T P \mathcal{N}_2(w_1, w_2),$$

with $\pi(w_1) = (\lambda + \Phi(\overline{\Pi}_1^{[1]}) w_1) w_1^2$ for $w_1 \geq 0$ and $\pi(w_1) = (\lambda + \Phi(\widetilde{\Pi}_1^{[1]}) w_1) w_1^2$ for $w_1 < 0$.

For $w_1 \in \Omega_1 \cup \Omega_2$, then $\pi(w_1) \leq -\alpha_3(||w_1||)$ according to (4). Hence

$$\begin{aligned} \dot{\mathcal{V}}(w_1, w_2) &< -\alpha_3(||w_1||) + w_1 \mathcal{N}_1(w_1, w_2) + w_2^T (P \bar{A}_2 + P \bar{A}_2) w_2 + 2 w_2^T P \mathcal{N}_2(w_1, w_2), \\ &\leq -(-\kappa_1 \nu + 1 + 2\kappa_2 \lambda_{max}(P)) ||w_2|| + \kappa_2 ||w_2|| - \kappa_2 ||w_2||, \\ &\leq -(-\kappa_1 \nu - \kappa_2 + 1 - 2\kappa_2 \lambda_{max}(P)) ||w_2||, \end{aligned}$$

with $\nu = \max_{\{w_1 : w_1 \in \Omega_1 \cup \Omega_2\}} ||w_1||$.

By choosing κ_1 and κ_2 such that $\kappa_1 \nu + \kappa_2 (1 + 2\lambda_{max}(P)) < 1$, then

$$\dot{\mathcal{V}}(w_1, w_2) < 0.$$

Hence, for $w_1 \in \Omega_1 \cup \Omega_2$, $\dot{\mathcal{V}}(w_1, w_2) < 0$. So, there exists a class $\mathcal{KL}$ function $\bar{\eta}$ such that

$$||w(t)|| \leq \bar{\eta}(||w(0)||, t).$$

When $w_1 \in \overline{\Omega}_1 \cup \overline{\Omega}_2$, and by considering w_1 as an input of the system

$$\dot{w}_2 = \bar{A}_2 w_2 + \mathcal{N}_2(w_1, w_2),$$

we deduce that $||w_2||$ is bounded, since $\bar{A}_2$ is Hurwitz. Hence, for $w_1 \in \overline{\Omega}_1 \cup \overline{\Omega}_2$, there exists $\bar{\varepsilon}$ such that

$$||w(t)|| \leq \bar{\varepsilon}.$$

From (4)-(4) we obtain

$$d_{\mathbb{B}_{\bar{\varepsilon}}}(w(t)) \leq \bar{\eta}(d_{\mathbb{B}_{\bar{\varepsilon}}}(w(0)), t).$$

So the origin of the whole dynamics is locally $\bar{\varepsilon}-$practically stable.

• Asymptotic Stabilization for $\lambda = 0$

In this case, generically, we have a transcontrollable bifurcation [17, 21]. Since $\varepsilon_1 = \varepsilon_2 = 0$, the sets $\overline{\Omega}_1$ and $\overline{\Omega}_2$ reduce to the origin. Hence, the origin of the reduced closed-loop system is asymptotically stable, since the solution converges to $\bar{\Omega}_1 \cup \bar{\Omega}_2 = \{0\}$. We deduce that the origin of the whole closed-loop dynamics is asymptotically stable by applying the center manifold theorem [5].

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