11

Low Codimension Control Singularities for Single Input Nonlinear Systems

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11.1 Introduction

Nonlinear dynamical systems exhibit complicated performance around bifurcation points. As the parameter of a system is varied, changes may occur in the qualitative structure of its solution around a point of bifurcation. In order to study dynamical systems with bifurcations, the following methodology is adopted in the theory of dynamical systems. First, the codimension of the bifurcation is computed, i.e. the smallest dimension of a parameter space which contains the bifurcation in a persistent way. Then, the system is embedded into a parameterized family of systems, with the number of parameters being equal to the codimension of the bifurcation. This family of systems, called miniversal deformation, describes the dynamics in the neighborhood of the bifurcation point. Finally, the dynamics of these systems is studied [6].

In this paper, we extend this methodology to controlled dynamical systems. Consider the class $C^k(\mathcal{X} \times \mathcal{U}, \mathbb{R}^n)$ of control systems of the form

$$\dot{x} = f(x, u) \tag{11.1}$$

where $x \in \mathcal{X}$, an open subset of $\mathbb{R}^n$, $u \in \mathcal{U}$, an open subset of $\mathbb{R}$ and f is C^k where $0 \leq k \leq \infty$. The equilibrium set $\mathcal{Z}$ of the control system is the set of all $(x^0, u^0) \in \mathcal{X} \times \mathcal{U}$ such that $f(x^0, u^0) = 0$. We are interested in studying control bifurcations, i.e., equilibria that are more difficult to stabilize than some of their neighboring equilibria.

A linear system of the form

$$\dot{x} = Fx + Gu \tag{11.2}$$

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is *controllable* if the smallest F invariant subspace containing the columns of G is $\mathbb{R}^n$. A controllable linear system can be steered from any state to any other state in any $t > 0$.

The *linear part* of the nonlinear system (11.1) around the equilibrium (x^0, u^0) is the system

$$\dot{x} = F(x - x^0) + G(u - u^0) \tag{11.3}$$

where

$$F = \frac{\partial f}{\partial x}(x^0, u^0), \qquad G = \frac{\partial f}{\partial u}(x^0, u^0).$$

If this linear system is controllable then the nonlinear system can be steered from any nearby state to any other nearby state in any $t > 0$.

The generic equilibrium of (11.1) has a controllable linear part so if an equilibrium is not linearly controllable then it is more difficult to control than some of its neighboring equilibria and hence a control singularity. The goal of this paper is to study and classify the low codimension control singularities of single input nonlinear control systems. Systems with multiple inputs will be treated in a longer version of this article.

The set of all equilibria of all control systems like (11.1) is infinite dimensional but the nature of control singularity frequently depends on the low degree terms of the Taylor series of f at the equilibrium. Therefore instead of studying the infinite dimensional object we study the Taylor series through degree k of all possible systems at all possible equilibria. The latter is called the system k-jet space which we introduce in the next section.

11.2 Equilibria in the k-Jet Space of Systems

The *system k-jet space* $\mathcal{S}^k(\mathcal{X} \times \mathcal{U}, \mathbb{R}^n)$ is the space of all tuples of the form

$$\Big((x, u), f(x, u), f^{(1)}(x, u), \ldots, f^{(k)}(x, u)\Big) \tag{11.4}$$

where $f \in C^k(\mathcal{X} \times \mathcal{U}, \mathbb{R}^n)$ and

$$f^{(j)}(x, u) = \frac{\partial^j f}{\partial (x, u)^j}(x, u).$$

Frequently when there is no chance of ambiguity we shall use the shortened notation $\mathcal{S}^k$. The terminology is a bit misleading, this is not a collection of systems but is a vector bundle with base $\mathcal{X} \times \mathcal{U}$ and fiber a real linear space of dimension $N(n, k)$ where

$$N(n, k) = n \begin{pmatrix} n + k + 1 \\ k \end{pmatrix}.$$

Moreover the notation (11.4) is very convenient but can be misleading. Each $f^{(j)}(x,u) = (f_1^{(j)}, \dots, f_n^{(j)})'$ and each $f_i^{(j)}$ is actually a symmetric tensor of degree j in $n+1$ indices. There is a natural projection of $\mathcal{S}^k$ onto $\mathcal{S}^l$ when $k \geq l \geq 0$.

A system (11.1) *realizes* a point in $\mathcal{S}^k$ if it has those derivatives at $\mathcal{X} \times \mathcal{U}$. Any point in $\mathcal{S}^k$ can be realized by a polynomial system of degree k but there are many other realizations.

It is more convenient to work with the systems jet space $\mathcal{S}^k(\mathcal{X} \times \mathcal{U}, \mathbb{R}^n)$ which is finite dimensional than the space of systems $C^k(\mathcal{X} \times \mathcal{U}, \mathbb{R}^n)$ which is infinite dimensional, particularly when studying a system locally around a particular (x,u) such as an equilibrium. The equilibrium set $\mathcal{Z}(f)$ of a system (11.1) is the set of all pairs $(x,u) \in \mathcal{X} \times \mathcal{U}$ such that $f(x,u) = 0$.

The equilibrium set $\mathcal{E}^k(\mathcal{X} \times \mathcal{U}, \mathbb{R}^n) \subset \mathcal{S}^k(\mathcal{X} \times \mathcal{U}, \mathbb{R}^n)$ is the space of all tuples of the form

$$\Big((x,u), 0, f^{(1)}(x,u), \dots, f^{(k)}(x,u)\Big). \tag{11.5}$$

Again when there is no chance of ambiguity we shall use the shortened notation $\mathcal{E}^k$. The equilibrium set $\mathcal{E}^k$ is also a vector bundle with base $\mathcal{X} \times \mathcal{U}$ and fiber a real linear space of dimension $N(n,k) - n$. Clearly $\mathcal{E}^k$ is a subbundle of $\mathcal{S}^k$ and it is carried onto $\mathcal{E}^l$, $0 \leq l \leq k$ by the natural projection. Notice that $\mathcal{Z}(f) \subset \mathcal{X} \times \mathcal{U}$ and depends on the system (11.1) but $\mathcal{E}^k \subset \mathcal{S}^k$ and we don't need a system to define it.

A k-jet in $\mathcal{E}^k$ is a *control singularity* it is more difficult to control than some of its neighboring tuples in $\mathcal{E}^k$. Our goal is to study the classes of control singularities that are of low codimension in $\mathcal{E}^k$. Due to space limitations, we will study only two control singularities, the fold and the transcontrollable singularities. Other control singularities will be studied in the longer version of this article.

There is another jet bundle that is of interest. The feedback k-jet bundle $\mathcal{K}^k$ is the set of all tuples of the form

$$\Big(x, \kappa(x), \kappa^{(1)}(x), \dots, \kappa^{(k)}(x)\Big) \tag{11.6}$$

where $\kappa : x \mapsto u$ is C^k mapping from $\mathcal{X}$ to $\mathcal{U}$ and

$$\kappa^{(j)}(x) = \frac{\partial^j \kappa}{\partial x^j}(x).$$

The maps $\kappa(x)$ are feedbacks and $\mathcal{K}^k$ is a fiber bundle with base $\mathcal{X}$ and fiber $\mathcal{U} \times \mathbb{R}^{M(n,k)}$ where

$$M(n,k) = \begin{pmatrix} n+k \\ k \end{pmatrix} - 1.$$

Given an equilibrium (x^0, u^0) of the system (11.1), a typical goal is to find a smooth feedback such that the closed loop system

$$\dot{x} = f(x, \kappa(x)) \tag{11.7}$$

$$u = \kappa(x) \tag{11.8}$$

is locally asymptotically stable to (x^0, u^0). Frequently the stability of the closed loop system can be decided by its k-jet at x^0 for small k. And the k-jet of the closed loop system can be computed from the k-jet of the system at (x^0, u^0) and the k-jet of the feedback at x^0 assuming that $\kappa(x^0) = u^0$.

Therefore we say an equilibrium k-jet with base point (x^0, u^0) is *stabilizable* if there exists a feedback k-jet with base point x^0 so that every realization of the former makes every realization of the latter locally asymptotically stable to (x^0, u^0).

11.3 Linear Feedback Group and Linear Normal Form

Let (x^0, u^0) be an equilibrium of the system (11.1), and let

$$F = \frac{\partial f}{\partial x}(x^0, u^0), \qquad G = \frac{\partial f}{\partial u}(x^0, u^0).$$

Then the controllability matrix of this pair is

$$\left[G \; FG \ldots F^{n-1}G \right].$$

This is an $n \times n$ matrix of rank r, with $0 \leq r \leq n$. The span of the columns of this matrix is an F invariant subspace of dimension r denoted by $\mathcal{V}$. If $r = n$ then the pair F, G is said to be *controllable* and the system (11.1) is said to be *linearly controllable* at (x^0, u^0).

If $r < n$, the system (11.1) is not linearly controllable at (x^0, u^0). Let $r_0 = n - r$, then r_0 is the number of state dimensions that can't be controlled by linear effects and r is the number of dimensions that can be controlled by the linear effects of u.

Now, consider a linear change of state coordinates and a linear feedback on (11.1)

$$\begin{aligned} x - x^0 &= Tz, \\ u - u^0 &= Kz + Lv, \end{aligned} \tag{11.9}$$

where T is an $n \times n$ invertible matrix, K is an $1 \times n$ matrix and L is a scalar. This change of coordinates and feedback is an element of the *linear feedback group* of the form

$$\begin{bmatrix} T & 0 \\ K & L \end{bmatrix} \tag{11.10}$$

This latter acts on the system (11.1) and so acts on its k-jets. Moreover, the transformation (11.9) takes the equilibrium (x^0, u^0) to the equilibrium at $(0, 0)$ of

$$\dot{z} = \bar{f}(z, v) = T^{-1} f(x^0 + Tz, u^0 + Kz + Lv). \tag{11.11}$$

This induces a mapping from $\mathcal{E}^k$ to $\mathcal{E}^k$. The k jet

$$\left((x^0, u^0), 0, \frac{\partial f}{\partial(x,u)}(x^0, u^0), \ldots, \frac{\partial^k f}{\partial(x,u)^k}(x^0, u^0)\right) \tag{11.12}$$

goes to

$$\left((0,0), 0, \frac{\partial \bar{f}}{\partial(z,v)}(0,0), \ldots, \frac{\partial^k \bar{f}}{\partial(z,v)^k}(0,0)\right) \tag{11.13}$$

If the linear part of the k-jet is $[F \;\; G]$ then it is changed to

$$\left[\bar{F}\; \bar{G}\right] = T^{-1}\left[F\; G\right]\begin{bmatrix} T & 0 \\ K & L \end{bmatrix}.$$

Given any system (11.1) there is an element of the linear feedback group which takes the linear part of the system into linear normal form. A system

$$\dot{z} = Az + Bv + O(z,v)^2 \tag{11.14}$$

is in *linear normal form* at the equilibrium $(z, v) = (0, 0)$ if

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & A_1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ B_1 \end{bmatrix} \tag{11.15}$$

where the $r_0 \times r_0$ matrix A_0 is in real Jordan form and the pair consisting of the $r \times r$ matrix A_1 and the $r \times 1$ matrix B_1 is in Brunovsky form.

The former means that A_0 is a block diagonal matrix with diagonal blocks of the form

$$\begin{bmatrix} \Lambda_1 & I & 0 & \ldots & 0 \\ 0 & \Lambda_2 & I & \ldots & 0 \\ & & \ddots & \ddots & \\ 0 & 0 & 0 & \ddots & I \\ 0 & 0 & 0 & \ldots & \Lambda_s \end{bmatrix} \tag{11.16}$$

where Λ_i is a scalar

$$\Lambda_i = a_i$$

or a 2×2 matrix of the form

$$\Lambda_i = \begin{bmatrix} a_i & -\omega_i \\ \omega_i & a_i \end{bmatrix}$$

with $\omega_i \neq 0$.

The latter means that

$$A_1 = \text{Diag} \begin{bmatrix} 0\,1\,0 \dots\, 0 \\ 0\,0\,1\, \dots\, 0 \\ \ddots \\ 0\,0\,0\, \dots\, 1 \\ 0\,0\,0\, \dots\, 0 \end{bmatrix}^{r \times r}$$

$$B_1 = \text{Diag} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^{r \times 1}$$

An equilibrium 1-jet in normal form is

$$\left((0,0), 0, \begin{bmatrix} A_0 & 0 & 0 \\ 0 & A_1 & B_1 \end{bmatrix} \right) \tag{11.17}$$

where A_0, A_1, B_1 are as above.

There is also a nonlinear feedback group that acts on (11.1). It includes the linear feedback group. It consists of C^k changes of state coordinates and state feedback of the form

$$\begin{aligned} x &= \theta(z) \\ u &= \kappa(z, v) \end{aligned} \tag{11.18}$$

where $(z, v) \mapsto (x, u)$ is a local diffeomorphism. This induces a corresponding action on points of $\mathcal{E}^k$. The nonlinear feedback group is stratified. There are near identity transformations of degree d which are of the form

$$\begin{aligned} x &= z + \theta^{[d]}(z) \\ u &= v + \kappa^{[d]}(z, v) \end{aligned} \tag{11.19}$$

where the superscript $^{[d]}$ indicates a polynomial vector field homogeneous of degree d. These do not form a subgroup as the composition of two such transformations typically has terms of degree d through d^2. A near identity transformation of degree d does not change the $d-1$ jet but it does modify the d-jet and higher jets. This allows one to bring the higher degree terms to normal form [4]

11.4 The Codimension of Orbits of the Linear Feedback Group

Control singularities are invariant under the linear and nonlinear feedback groups. If the original system (11.1) has a control singularity at an equilibrium

(x^0, u^0) then the transformed system has the same type of control singularity at the transformed equilibrium.

A class of *linear control singularities* is most conveniently defined by conditions on the 1 jet of the system in normal form. For such singularities, we are only interested in the action of the linear feedback group. The fold $\mathcal{F}$, treated in section 11.5.2, is a linear control singularity.

To compute the codimension of a class of linear control singularities we proceed as follows. As we said before the singular class is most conveniently defined by certain conditions on the linear normal form at the equilibrium $(0, 0)$. The normal form may depend on one or more parameters. All other elements of the singular class are obtained by a linear feedback transformation acting on a singularity in normal form.

Hence we must study the action of the linear feedback group

$$z = Tx \tag{11.20}$$

$$v = Kx + Lu \tag{11.21}$$

on systems in linear normal form (11.14). We partition (11.10) compatibly with (11.15)

$$\begin{bmatrix} T & 0 \\ K & L \end{bmatrix} = \begin{bmatrix} T_{00} & T_{01} & 0 \\ T_{10} & T_{11} & 0 \\ K_0 & K_1 & L \end{bmatrix} \tag{11.22}$$

The result is a new system

$$\dot{x} = Fx + Gu + O(x, u)^2$$

where

$$\begin{bmatrix} F & G \end{bmatrix} = T^{-1} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} T & 0 \\ K & L \end{bmatrix}$$

$$\begin{bmatrix} F_{00} & F_{01} & G_0 \\ F_{10} & F_{11} & G_1 \end{bmatrix} = \begin{bmatrix} S_{00} & S_{01} \\ S_{10} & S_{11} \end{bmatrix} \begin{bmatrix} A_1 & 0 & 0 \\ 0 & A_1 & B_1 \end{bmatrix} \begin{bmatrix} T_{00} & T_{01} & 0 \\ T_{10} & T_{11} & 0 \\ K_0 & K_1 & L \end{bmatrix}$$

with $S = T^{-1}$.

Tannenbaum [5] has descibed the action of the group of linear changes of state coordinates, i. e. $K_1 = 0, L = I$, acting on linearly controllable systems, i.e. $r_0 = 0$. But we are interested in the full feedback group acting on possibly linearly uncontrollable systems.

When computing the codimension of an orbit of this action, it is simpler to compute the codimension of the infinitesmal action which is a linear calculation. Consider a curve $T = T(\mu),\ K = K(\mu),\ L = L(\mu)$ in the linear feedback group parameterized by $\mu \in I\!R$ where $T(0) = I,\ K(0) = 0,\ L(0) = I$. Let $'$ denote differentiation with respect to μ at $\mu = 0$ then

$$\begin{bmatrix} F_{00} & F_{01} & G_0 \\ F_{10} & F_{11} & G_1 \end{bmatrix}' = \left(T^{-1} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} T & 0 \\ K & L \end{bmatrix} \right)'$$

$$= \begin{bmatrix} A_0 T'_{00} - T'_{00} A_0 & A_0 T'_{01} - T'_{01} A_1 & -T'_{01} B_1 \\ A_1 T'_{10} - T'_{10} A_0 + B_1 K'_0 & A_1 T'_{11} - T'_{11} A_1 + B_1 K'_1 & -T'_{11} B_1 + B_1 L' \end{bmatrix}$$

This action splits into four mappings

$$T'_{00} \mapsto F'_{11} = A_0 T'_{00} - T'_{00} A_0, \tag{11.23}$$

$$T'_{01} \mapsto \begin{bmatrix} F_{01} & G_0 \end{bmatrix}' = \begin{bmatrix} A_0 T'_{01} - T'_{01} A_1, & -T'_{01} B_1 \end{bmatrix}, \tag{11.24}$$

$$\begin{bmatrix} T'_{10} \\ K'_0 \end{bmatrix} \mapsto F'_{21} = A_1 T'_{10} - T'_{10} A_0 + B_1 K'_0, \tag{11.25}$$

$$\begin{bmatrix} T'_{11} & 0 \\ K'_1 & L' \end{bmatrix} \mapsto \begin{bmatrix} F_{11} & G_1 \end{bmatrix}' = \begin{bmatrix} A_1 T'_{11} - T'_{11} A_1 + B_1 K'_1, & -T'_{11} B_1 + B_1 L' \end{bmatrix}. \tag{11.26}$$

Each mapping is from a real vector space to another. For each of them, we wish to compute the codimension of its range and find a maximal set of linearly independent vectors which are transverse to the range.

The first linear mapping (11.23) is the action of infinitesmal linear changes of the uncontrollable coordinates. It goes from $\mathbb{R}^{r_0^2}$ to $\mathbb{R}^{r_0^2}$. This is the same mapping that occurs when studying dynamical systems without controls. It is never an isomorphism and the codimension of its range depends on A_0. An analysis of this map can be found in Wiggins [6] on page 315. We state the results for the cases where A_0 is 1×1 or 2×2.

If A_0 is 1×1 then the map (11.23) is identically zero so the range is of codimension one. A 1×1 matrix transverse to the range is 1.

If A_0 is 2×2 then there are several possibilities. We enumerate those of codimension two.

If A_0 has distinct, nonzero real eigenvalues then the range of (11.23) is of codimension two. Two matrices transverse to the range are

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

If A_0 has two complex eigenvalues whose real and imaginary parts are both nonzero then the range of (11.23) is of codimension two. Two matrices transverse to the range are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

If A_0 is a 2×2 Jordan block

$$A_0 = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}$$

where $a \neq 0$ then the codimesion is also two. Two matrices transverse to the range are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Next consider the linear mapping (11.24). This is a mapping from $I\!R^{r_0 r}$ to $I\!R^{r_0(r+1)}$ which is clearly not onto, but it is one to one. Let $X = T'_{01}$ and $X_{\cdot,j}$ denote the j^{th} column of the matrix X. Suppose

$$\begin{aligned} A_0 X - X A_1 &= 0 \\ X B_1 &= 0 \end{aligned}$$

then these equations become

$$\begin{aligned} A_0 X_{\cdot,j} - X_{\cdot,j-1} &= 0 \\ X_{\cdot,r} &= 0 \end{aligned}$$

so $X = 0$. Therefore the range of the mapping (11.24) has codimension r_0. One choice of r_0 independent $r_0 \times (r+1)$ matrices transverse to the range is

$$\begin{bmatrix} F_{01} \; G_0 \end{bmatrix} = \begin{bmatrix} \mathbf{e_i} \; 0 \ldots 0 \end{bmatrix}, \quad i = 1, \ldots, r_0 \tag{11.27}$$

where $\mathbf{e_i}$ is the i^{th} vector in $I\!R^{r_0}$.

Next consider the linear mapping (11.25). This is a mapping from $I\!R^{r r_0 + r_0}$ to $I\!R^{r r_0}$ which we now show is onto. Let $X = T'_{10}$ and $X_{i,\cdot}$ denote the i^{th} row of the matrix X. For $1 \leq i \leq r-1$

$$\begin{aligned} (A_1 X - X A_0 + B_1 K'_0)_{i,\cdot} &= X_{i+1,\cdot} - X_{i,\cdot} A_0 \\ (A_1 X - X A_0 + B_1 K'_0)_{n,\cdot} &= -X_{n,\cdot} A_0 + K'_1 \end{aligned}$$

Hence we choose $X_{1,\cdot}$ arbitrarily, then for for $2 \leq i \leq r$ we choose $X_{i,\cdot}$ to arbitrarily fix the $i-1^{th}$ row and K'_1 to arbitrarily fix the r^{th} row.

Finally we look at the linear mapping (11.26). This is a mapping from $I\!R^{r^2+r+1}$ to $I\!R^{r^2+r}$. If the controllability index $(0, r)$ is generic in the space of systems with r states and one input then this map is onto. If the pair is not generic then we must do a case by case study.

11.5 Versal Deformations and Low Codimension Linear Control Singularities of Scalar Input Sytems

11.5.1 Versal Deformations

Given a class of control singularities $\mathcal{G} \in \mathcal{E}^k$, one would like to study the types of equilibrium k jets that can be obtained by small perturbations. A family of control singularities is always invariant under the action of the feedback group.

Let $\mathcal{G} \subset \mathcal{E}^k$ which is invariant under the action of the feedback group. A C^k *versal deformation* of $\mathcal{G}$ is a C^k parametrized subset of $\mathcal{E}^k$ of the form

$$\begin{array}{l}\phi : \mathcal{P} \to \mathcal{E}^k \\ \phi : \mu \mapsto \phi(\mu) = ((x,u), 0, \phi^{(1)}(\mu), \dots, \phi^{(k)}(\mu))\end{array} \tag{11.28}$$

defined for μ in some neighborhood $\mathcal{P}$ of $0 \in I\!\!R^p$, which intersects $\mathcal{G}$ at $\mu = 0$ and which is transversal to $\mathcal{G}$. The versal deformation is said to be *miniversal* if the dimension p is minimal among all versal deformations. The minimal p is the *codimension* of $\mathcal{G}$.

A C^k *versal feedback* for a versal deformation (11.28) is a mapping

$$\begin{array}{l}\psi : \mathcal{P} \to \mathcal{K}^k \\ \psi : \mu \mapsto \psi(\mu) = ((x,u), 0, \psi^{(1)}(\mu), \dots, \psi^{(k)}(\mu)).\end{array} \tag{11.29}$$

A versal feedback is *stabilizing* if at each $\mu \in \mathcal{P}$, $\psi(\mu)$ stabilizes $\phi(\mu)$.

11.5.2 Fold Control Singularities

Consider the scalar input system

$$\dot z_0 = a z_0 + O(z,v)^2 \tag{11.30}$$

$$\dot z_1 = A_1 z_1 + B_1 v + O(z,v)^2 \tag{11.31}$$

where the $z_0 \in I\!\!R,\ z_1 \in I\!\!R^{n-1}$ and

$$\begin{aligned} A_0 &= a \neq 0 \\ A_1 &= \begin{bmatrix} 0\,1\,0 \dots & 0 \\ 0\,0\,1 \dots & 0 \\ & \ddots \\ 0\,0\,0 \dots & 1 \\ 0\,0\,0 \dots & 0 \end{bmatrix}^{r\times r} \\ B_1 &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}^{r\times 1} . \end{aligned} \tag{11.32}$$

Its 1-jet at the origin is

$$\left((0,0), 0, \begin{bmatrix} a & 0 & 0 \\ 0 & A_1 & B_1 \end{bmatrix}\right) \tag{11.33}$$

This equilibrium 1-jet is the simplest example of a control singularity and is called a fold. The subset $\mathcal{F} \subset \mathcal{E}^k$ of *fold singularities* is the set of all 1-jets of the form

$$\left((x^0, u^0), 0, \begin{bmatrix} F_{00} & F_{01} & G_0 \\ F_{10} & F_{11} & G_1 \end{bmatrix}\right) \tag{11.34}$$

whose normal form is (11.33) for some $a \neq 0$. We shall show that $\mathcal{F}$ is of codimension one in $\mathcal{E}^k$.

Let us study the infinitesmal action of the linear feedback group on A, B (11.32). The range of the linear mapping (11.24) is of codimension one so from (11.27) we obtain a nonzero $n \times (n+1)$ matrix transverse to the orbit of A, B under the linear feedback group. The linear mapping (11.23) is identically zero so there is another linearly independent $n \times (n+1)$ matrix transverse to the orbit of A, B under the linear feedback group.

But perturbations in one of these directions can be accomplished by varying a and so it is not transverse to $\mathcal{F}$. Therefore $\mathcal{F}$ is a codimension one set of control singularities and a miniversal deformation of it is

$$\mu \mapsto \begin{bmatrix} F(\mu) & G(\mu) \end{bmatrix}$$

$$F(\mu) = \left[\begin{array}{c|c} a & \mu\ 0\ 0\ \dots\ 0 \\ \hline 0 & 0\ 1\ 0\ \dots\ 0 \\ 0 & 0\ 0\ 1\ \dots\ 0 \\ & \ddots \\ 0 & 0\ 0\ 0\ \dots\ 1 \\ 0 & 0\ 0\ 0\ \dots\ 0 \end{array}\right], \quad G(\mu) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \\ 0 \\ 1 \end{bmatrix}$$

Now we see why the fold terminology. The controllabilty matrix of the versal deformation in reverse order is

$$\begin{bmatrix} F(\mu)^{n-1}G(\mu) & \dots & F(\mu)G(\mu) & G(\mu) \end{bmatrix} = \left[\begin{array}{c|c} \mu & 0 \\ \hline 0 & I \end{array}\right].$$

Notice that all these 1-jets are controllable except for $\mu = 0$ and the controllability reverses orientation (folds over) at $\mu = 0$.

There are two subclasses of fold singularities, those where $a < 0$ and those with $a > 0$. There is an important distinction between these subclasses. The former are linearly stabilizable, i.e., there exists a linear feedback $u = Kx$ so that all the poles of the linear part of the closed loop dynamics $A + BK$ are in the left half plane. In fact the versal deformation of a fold singularity with $a < 0$ is versally stabilizable by a versal linear feedback of the form

$$((x, u), 0, [K_0, K_1])$$

where $K_0 = 0$, and K_1 is such that $A_1 + B_1 K_1$ is Hurwitz.

But if $a > 0$ then the closed loop dynamics will always have at least one unstable eigenvalue a. The stabilization problem of systems with fold control singularities when $a > 0$ was treated in [2].

11.5.3 Transcontrollable Singularities

A transcontrollable singularity is a degenerate fold where $a = 0$. This means that the stabilizability of a system realizing this 1-jet is decided by its higher order terms. The class of transcontrollable singularities denoted by $\mathcal{TC}$ is of codimension two and a versal deformation of it is

$$(\mu_1, \mu_2) \mapsto \left[\, F(\mu_1, \mu_2)\; G(\mu_1, \mu_2) \,\right]$$

$$F(\mu_1, \mu_2) = \left[\begin{array}{c|cccccc} \mu_1 & \mu_2 & 0 & 0 & \ldots & 0 \\ \hline 0 & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 \\ & & & & \ddots & \\ 0 & 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 \end{array}\right], \quad G(\mu_1, \mu_2) = \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \\ 0 \\ 1 \end{array}\right]$$

Notice that a transcontrollable singularity can be perturbed into a fold by changing μ_1 from zero and can be perturbed into a linearly controllable 1-jet by changing μ_2 from zero.

The stabilization problem of systems with transcontrollable singularities was treated in [3].

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