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Some results on inverse optimality based designs

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Abstract

Our objective in this paper is to give some results on inverse optimal designs in view of robustness to known/unknown, but ignored input dynamics. This problem comes from the presence of actuators or the wish for using simplified models. Stabilizing control laws may not be robust to this type of uncertainties. By exploiting the robustness of optimal controllers, "domination redesign" of the control Lyapunov function based on Sontag's formula has been shown to possess robustness to static and dynamic input uncertainties. In this paper we provide some results on designs based on inverse optimality.

Keywords: Inverse optimality; Control Lyapunov functions; Robustness; Ignored input dynamics

1. Introduction

Many existing nonoptimal nonlinear design methods are based on the notion of control Lyapunov function (CLF). This function is defined to be C^1, positive definite, proper, and whose derivative along $f(x)+g(x)u$ can be made negative for each $x \neq 0$ by an appropriate choice of the input u. When a CLF for $\dot{x}=f(x)+g(x)u$ is given, a C^0 controller $k(x)$ can be constructed in such a manner that the derivative of the CLF along $\dot{x}=f(x)+g(x)k(x)$ is strictly negative whenever $x \neq 0$. Different types of CLF's are appropriate for different tasks (e.g., robust stabilization, adaptive tracking), and if a CLF is known, the construction of globally asymptotically stabilizing controllers is straightforward [1,15]. There are several methods available for constructing CLF's for certain relevant classes of nonlinear systems, including feedback linearization, backstepping and forwarding [14]. However, these CLF methods do not guarantee adequate controller performance.

Various attempts have been made to improve the performance of CLF design methods. In particular, inverse optimal methods generate controllers which optimize meaningful cost functionals. Inverse optimality is a notion introduced for linear systems by Kalman [9], and for nonlinear systems by Moylan and Anderson [11]. It was studied later by Freeman and Kokotović [3] (for further reading, please see [4,5,8,17] and the references therein). It arises from the wish to have, for nonlinear systems, the beneficial property of robustness of optimal controllers without resolving the Hamilton–Jacobi–Bellman equation, which is not always a feasible task. In the inverse optimal approach, a stabilizing feedback is designed first, and then is shown to be optimal for a cost functional of the form

$$\int_0^\infty \{\ell(x)+r(x,u)\}\,\mathrm{d}t,$$

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where l and r are positive functions, with $r(x,0)=0$, and for each fixed x, $r(x,u)=\xi_x(\|u\|_x)$ for some convex class $\mathscr{K}$ function ξ_x and some norm $\|\cdot\|_x$. In other words, the functions l and r are a posteriori determined from the data of a particular stabilizing controller, rather than a priori chosen by the designer, regardless of the control law.

This inverse optimal result is sufficient to establish robustness to static uncertainties. The dependence of r on x renders impossible to guarantee robustness against dynamic uncertainties. Domination redesign was introduced in [14] where $r(x,u)=u^{\mathrm{T}}R(x)u$, to suppress the dependence of r on x through reassignation of the levels of the CLF and hence permits to have a stability margin against input unknown dynamics.

In this paper we use a new characterization of CLF's given in [6], in order to solve an inverse optimal design problem. The paper is organized as follows: in Section 2 we recall some results on the characterization of CLF's. In Section 3 we show that by deforming a CLF we obtain a Bellman function for criteria where the penalty on the control is not convex, but minorized by an affine function. Then, we solve an inverse optimality problem.

2. A characterization of control Lyapunov functions

Consider the system

$$\dot{x}=f(x)+g(x)u, \quad x\in\mathbb{R}^n, \; u\in\mathbb{R}^m. \tag{1}$$

Let Γ be the set of functions $\gamma\in C^0(\mathbb{R}^p,\mathbb{R}_+)$, such that

- $\gamma(0)=0$ and for all $s\in\mathbb{R}^p\backslash\{0\}$, $\gamma(\varphi s)/\varphi$ is an increasing function of φ,
- for all $s\in\mathbb{R}^p\backslash\{0\}$, $\lim_{\varphi\to+\infty}\gamma(\varphi s)/\varphi=+\infty$.

The following theorem, proved in [6], gives a characterization of CLF's for systems (1).

Theorem 2.1. *Let V be a positive definite and proper function. Then V is a CLF if and only if $\forall\gamma\in\Gamma$ and $\forall\varepsilon>0$, $\exists\psi_\varepsilon\in C^1(\mathbb{R}_+,\mathbb{R}_+)$, with ψ'_ε positive definite, such that $\lim_{\xi\to+\infty}\psi_\varepsilon(\xi)=+\infty$. And we have*

$$L_f\psi_\varepsilon(V(x))\langle\gamma(L_g\psi_\varepsilon(V(x))); \;\forall|x|\rangle\varepsilon. \tag{2}$$

Moreover, we can take ψ_ε independent of ε if and only if γ is such that

$$\exists k>0, \quad \limsup_{x\to 0}\frac{L_fV(x)}{\gamma(kL_gV(x))}<\frac{1}{k}. \tag{3}$$

Remarks. (1) In this theorem, we have to resolve inequality (2) for ψ_ε, and that for a given $\gamma\in\Gamma$.

(2) Concerning the independence of ψ_ε on ε. Condition (3) have not to be satisfied for all $\gamma\in\Gamma$. We have to find function(s) $\gamma_0\in\Gamma$ such that (3) is satisfied; for these functions, ψ_ε is independent on ε and permits to (2) to be valid everywhere.

A direct application of Theorem 2.1 to the stabilization problem of (1) permits to deduce a stabilizing controller.

Theorem 2.2. *Let V be a CLF and $\gamma\in\Gamma$ be a function satisfying properties* (i) *to* (iii) *and which satisfies condition* (3). *Let the control be*

$$u(x)=-\frac{\gamma(L_g\psi(V(x)))}{|L_g\psi(V(x))|}\cdot\frac{L_gV(x)^{\mathrm{T}}}{|L_gV(x)|} \quad \text{if } L_gV(x)\neq 0,$$

$$\in\left\{u\colon |u|\leqslant\sup_{\|v\|=1}\gamma(v)\right\} \quad \text{if } L_gV(x)=0. \tag{4}$$

Then, for all x there exists $T \leqslant +\infty$ and an absolutely continuous function $\phi(x,t)$ on $[0,+T)$, called solution, such that

$$\dot{\phi}(x,t) = f(\phi(x,t)) + \dot{g}(\phi(x,t))u(\phi(x,t)) \quad \textit{for almost all } t \in [0,T),$$
$$\phi(x,0) = x \tag{5}$$

and the origin is strongly[1] *asymptotically stable.*

Proof. See Appendix A.2. □

Remarks. (1) Continuity of the controller at the origin is guaranteed if γ satisfies $\lim_{|u|\to 0}\gamma(u)/|u| = 0$.
(2) If (3) does not hold, then ψ depends on ε and the controller will be strongly practically stabilizing system (1).

3. Inverse optimality

In this section we prove that $\psi_\varepsilon(V(x))$ is a Bellman function associated to a non-quadratic criterion not necessarily convex. Then, we prove that (4) solves an inverse optimality problem.

The case $\gamma(s) = |s|^2$ of Theorem 2.1 is in fact already known. Indeed, in [14,10] the authors prove that if V is a CLF and satisfies a condition at the origin, then there exists a C^1 function ψ such that $\psi(V(x))$ is the optimal value function associated to the cost functional

$$J(x) = \int_0^\infty \left[\ell(x) + \frac{1}{4}|u|^2\right] \mathrm{d}t$$

with ℓ a positive definite function. More precisely, $\psi(V(x))$ is a solution of the following Hamilton–Jacobi–Bellman (HJB) equation:

$$\ell(x) + L_f\psi(V(x)) + \min_u\{L_g\psi(V(x))u + \tfrac{1}{4}u^2\} = 0, \tag{6}$$

i.e.,

$$\ell(x) + L_f\psi(V(x)) - |L_g\psi(V(x))|^2 = 0.$$

Our result with $\gamma(s) = |s|^2$, follows since ℓ is positive definite. This yields

$$L_f\psi(V(x)) < |L_g\psi(V(x))|^2.$$

Theorem 2.1 allows to prove the following relation between CLF and optimal value functions.

Theorem 3.1. *Let V be a CLF and r be a positive function such that*[2]

$$\lim_{\|u\|\to\infty} \frac{r(u)}{\|u\|} = \infty \tag{7}$$

and there exists $(r_0, r_1) \in \mathbb{R}^m \times \mathbb{R}$ *such that* $r(u) \geqslant r_0^{\mathrm{T}}u + r_1$. (8)

Then, for all $\varepsilon > 0$ there exists a function ψ_ε and a positive definite function ℓ_ε, such that $\psi_\varepsilon(V(x))$ solves the following Hamilton–Jacobi–Bellman equation for all $|x| \geqslant \varepsilon$.

$$\ell(x) + L_f\psi_\varepsilon(V(x)) + \min_u\{L_g\psi_\varepsilon(V(x))u + r(u)\} = 0. \tag{9}$$

[1] The origin is strongly asymptotically stable if, for each $\varepsilon > 0$, there exists $\delta > 0$ which possesses the following property: For each x, such that $|x| < \delta$, each solution $\phi(x,t)$ exists on $[0,+\infty)$, satisfying the inequality $|\phi(x,t)| < \varepsilon$ and $\phi(x,t) \to 0$ as $t \to \infty$.

[2] Condition (8) is weaker than convexity. This condition requires the knowledge of only *one* affine function minirozing r. While a convex function is the supremum of a *family* of affine functions; hence any convex function is minorized by some affine function [7,13].

Before proving Theorem 3.1, we need the following lemma:

Lemma 3.1. *If r satisfies* (7) *then* $\sup_u\{su - r(u)\} < +\infty$ *for all* $s \in \mathbb{R}^p$.

Proof. Condition (7) implies

$$\forall \varepsilon > 0,\ \exists \delta(\varepsilon) > 0\colon \|u\| \geqslant \delta(\varepsilon) \Rightarrow \varepsilon\|u\| \leqslant r(u).$$

Then,

$$\|u\| \geqslant \delta(\varepsilon)\colon su - r(u) \leqslant (\|s\| - \varepsilon)\|u\|.$$

For all $\varepsilon > 0$ and for $\|s\| \leqslant \varepsilon$, we have

$$su - r(u) \leqslant (\|s\| - \varepsilon)\|u\| \leqslant (\|s\| - \varepsilon)\delta(\varepsilon) < +\infty.$$

Hence, for all $\varepsilon > 0$

$$\sup_{\|s\| \leqslant \varepsilon} \{su - r(u)\colon \|u\| \geqslant \delta(\varepsilon)\} < +\infty.$$

On the other hand for $\|u\| \leqslant \delta(\varepsilon)$, and since r is positive definite, we have

$$su - r(u) \leqslant su \leqslant \|s\|\|u\| \leqslant \|s\|\delta(\varepsilon). \tag{10}$$

which implies that

$$\sup\{su - r(u)\colon \|u\| \leqslant \delta(\varepsilon)\} < +\infty. \qquad \square$$

Proof of Theorem 3.1. Let V be a CLF, from Theorem 2.1 there exists ψ_ε with ψ'_ε positive definite, such that (2) is satisfied. Moreover, according to Lemma 3.1, the function

$$\mathscr{F}(r)(s) = \sup_{s^*}\{\langle s^*, s\rangle - r(s^*)\}, \quad s \in \mathbb{R}^m, \tag{11}$$

called the Legendre–Fenchel transform of r is bounded[3] on $\mathbb{R}^m$. We claim that

$$\gamma(s) = \mathscr{F}(r)(-s) \tag{12}$$

is a function which satisfies the conditions of Theorem 2.1. Indeed,

- γ is positive definite: From (ii) in Lemma A.1, $\gamma(s) = \mathscr{F}(r)(-s) \geqslant -r(0)$. Moreover, $r(0) = 0$ hence $\gamma(s) \geqslant 0$.
- $\gamma(0) = 0$: We have $\gamma(0) = \mathscr{F}(r)(0) = -\inf_{s^*} r(s^*)$. Since r is positive definite, (8) is satisfied, and $r(0) = 0$, then $-\inf_{s^*} r(s^*) = 0$, which shows that $\gamma(0) = 0$.
- $\gamma(\varphi s)/\varphi$ is an increasing function of φ and $\lim_{\varphi\to\infty} \gamma(\varphi s)/\varphi = \infty$: Since the Legendre–Fenchel transform is convex, then for $0 < \varphi_1 < \varphi_2$, we have

  $$\mathscr{F}(r)(\varphi_1 s) \leqslant \frac{\varphi_1}{\varphi_2}\mathscr{F}(r)(\varphi_2 s) + \frac{(\varphi_2 - \varphi_1)}{\varphi_2}\mathscr{F}(r)(0) = \frac{\varphi_1}{\varphi_2}\mathscr{F}(r)(\varphi_2 s).$$

  This proves monotonicity. Moreover, $\lim_{\varphi\to\infty} \gamma(\varphi s)/\varphi = \infty$, since $\mathscr{F}(r)$ satisfies (iii).

  Since γ in (12) is in Γ, then from Theorem 2.1

  $$\forall \varepsilon > 0,\ \exists \psi_\varepsilon \in C^1(\mathbb{R}_+, \mathbb{R}_+)\colon L_f\psi_\varepsilon(V(x)) < \mathscr{F}(r)(-L_g\psi_\varepsilon(V(x))); \quad \forall |x| \geqslant \varepsilon, \tag{13}$$

this equation can be put in the following form:

$$\ell_\varepsilon(x) + L_f\psi_\varepsilon(V(x)) - \mathscr{F}(r)(-L_g\psi_\varepsilon(V(x))) = 0, \quad \forall |x| \geqslant \varepsilon \tag{14}$$

with $\ell_\varepsilon(x)$ being positive definite, since inequality (13) is strict.

[3] The boundedness of the Legendre–Fenchel transform is guaranteed for all s such that $\|s\| \leqslant \|r_0\|$, if (8) is satisfied. When (7) is also satisfied then boundedness is guaranteed for all $s \in \mathbb{R}^m$ [7].

Eq. (14) can be put in the following form:

$$\ell_\varepsilon(x) + L_f\psi_\varepsilon(V(x)) + \min_u\{L_g\psi_\varepsilon(V(x))u + r(u)\} = 0, \quad \forall|x| \geqslant \varepsilon, \tag{15}$$

which corresponds to (9).

This shows that the CLF $\psi_\varepsilon(V(x))$ is also a Bellman function. □

If (3) is satisfied, then (9) is satisfied everywhere. This condition have been used in [14,10] (see also [12]) for the case $\gamma(s) = |s|^2$. As proved in [6] (see also [16]), this property is related to the small control property. [4] Let us recall that this property guarantees the smoothness of a controller at the origin. In our case, this property allows to the HJB equation to be satisfied everywhere.

Now let us treat the inverse optimality problem (i.e. we show that controller (4) minimizes a cost function).

Let us consider that our cost functionals are characterized by functions ℓ and r, such that

H1. ℓ is a positive definite function.

H2. r is a positive definite continuous function with $r(0) = 0$, and such that there exists an affine function minorizing r on $\mathbb{R}^m$, i.e. $\exists(r_0, r_1) \in \mathbb{R}^m \times \mathbb{R}$ such that $r(u) \geqslant r_0^T u + r_1$ for all u.

Using such a pair (ℓ, r) we consider a cost functional J to be minimized by a control law. Let the cost [5] be defined by

$$J(x) = \int_0^\infty [\ell(X(x,t,u)) + r(u(t))]\,\mathrm{d}t \tag{16}$$

associated to (1), where $X(x,t,u)$ is a solution of (1) under the control function $u(t)$ and starting from x.

Theorem 3.2. *Consider system* (1) *with controller* (4), *and a function* $\gamma \in \Gamma$ *minorized by an affine function on* $\mathbb{R}^m$ (*i.e.* $\exists s_0 \in \mathbb{R}^m$ *such that* $\gamma(s) \geqslant s_0^{\mathrm{T}} s$), *and which satisfies condition* (3). *Then there exists a pair* (ℓ, r) *satisfying hypotheses* H1 *and* H2 *such that* (16) *is minimized.*

Proof of Theorem 3.2. As stated in Theorem 2.1, V is a CLF if and only if for all $\gamma \in \Gamma$ there exists ψ_ε such that (2) is satisfied. Hence, a CLF is characterized by γ and ψ_ε. The problem is to prove that the CLF satisfies a HJB equation, for that we use a property (iv) of Lemma A.1.

In this case, we define r as

$$r(u) = \sup_s(-su - \gamma(s)) = \mathscr{F}(\gamma)(-u),$$

This transformation is bounded on $\mathbb{R}^m$ since $\lim_{\varphi\to+\infty} \gamma(\varphi s)/\varphi = +\infty$ (apply Lemma 3.1 to γ). This property is called co-finiteness.

Let the cost be defined by

$$J = \int_0^\infty \{\ell(x) + \mathscr{F}(\gamma)(-u))\}\,\mathrm{d}t. \tag{17}$$

The function defined by $\ell(x) = -L_f V(x) + \mathscr{F}^*(L_g V(x))$, with $\mathscr{F}^*(s) = \sup_u(-su - \mathscr{F}(\gamma)(-u))$, is positive definite. For that we have to prove that $\mathscr{F}^*(s) \in \Gamma$. This property is satisfied due to the fact that $\gamma(s)$ being minorized by an affine function, $\mathscr{F}(\gamma)(s)$ is also minorized by an affine function ($\mathscr{F}(\gamma)(s)$ is the supremum of a set of affine functions). This shows that $\mathscr{F}^*(s)$ is the Legendre–Fenchel transform of $\mathscr{F}(\gamma)$. Since this transform is convex and co-finite then $\mathscr{F}^*(s) \in \Gamma$.

[4] A C^1 function V satisfies "the small control property" if there exists a continuous, positive definite function $\Omega(x)$ such that, for each $\varepsilon > 0$, there exists $v > 0$ such that, for all x meeting $|x| \leqslant v$, we can find $u(x)$ satisfying $|u| \leqslant \varepsilon$ and

$$L_f V(x) + L_g V(x)u(x) + \Omega(x) \leqslant 0.$$

[5] The function r could depend on x. However it must be constrained as follows: for a fixed x the function $r(x,u) = \Theta_x(\|u\|_x)$ is minorized by an affine function of u, for some norm $\|\cdot\|_x$. This condition relaxes the convexity condition usually required.

A particular case of this result is obtained when $r \in \overline{\mathrm{Conv}}\,\mathbb{R}^m$ (i.e. we require that r must be a closed convex function). In this case we have $\mathscr{F}(\mathscr{F}(r))(u) = r(u)$ (use property (vi) of Lemma A.1). If V is a Lyapunov function which satisfies conditions of Theorem 2.1, then the control Lyapunov function $\psi(V(x))$ for (1) is also a Bellman function for the same system, with (17) as criterion. This follows from the fact that $\gamma(u) = \mathscr{F}(r)(-u)$ and $\mathscr{F}(\mathscr{F}(r))(u) = r(u)$ if and only if $r \in \overline{\mathrm{Conv}}\,\mathbb{R}^n$. Moreover, (2) implies that $\ell(x) = -L_f\psi(V(x)) + \gamma(L_g\psi(V(x)))$ is a positive definite function. □

This link with inverse optimality shows, that if γ is minorized by some affine function of s, then $\mathscr{F}(\gamma)$ is also a function which satisfies the conditions of Theorem 2.1.

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Appendix A

A.1. Some properties of the Legendre–Fenchel transform

Lemma A.1. *Let $\chi : \mathbb{R}^n \to \mathbb{R}$ be a function such that*

$$\text{for some } (s_0, b) \in \mathbb{R}^n \times \mathbb{R},\ \chi(s) \geqslant \langle s_0, s\rangle - b, \quad \forall s \in \mathbb{R}^n.$$

Then, the Legendre–Fenchel transform $\mathscr{F}(\chi)(s) = \sup_{s^*}\{\langle s^*, s\rangle - \chi(s^*)\}$ for $s \in \mathbb{R}^m$ satisfies the following properties:

(i) *$\mathscr{F}(\chi)(s)$ is a convex function (i.e., for $\alpha \in (0,1)$: $\mathscr{F}(\chi)(\alpha s_1 + (1-\alpha)s_2) \leqslant \alpha\mathscr{F}(\chi)(s_1) + (1-\alpha)\mathscr{F}(\chi)(s_2)$).*
(ii) *Let $\tilde{\chi}$ be a function satisfying the same properties of χ and such that $\tilde{\chi}(s) \leqslant \chi(s)$ then $\mathscr{F}(\chi)(s) \geqslant \mathscr{F}(\tilde{\chi})(s)$.*
(iii) *$\mathscr{F}(\mathscr{F}(\chi))(s) \leqslant \chi(s)$.*
(iv) *$\lim_{\varphi\to\infty} \mathscr{F}(\chi)(\varphi s)/\varphi = \infty$.*
(v) *$\mathscr{F}(\chi)(0) = -\inf_{s^*}\{\chi(s^*)\}$ and $\inf_{s^*}\mathscr{F}(\chi)(s^*) = -\mathscr{F}(\mathscr{F}(\chi))(0) \geqslant -\chi(0)$.*
(vi) *$\mathscr{F}(\mathscr{F}(\chi)) = \overline{\mathrm{Conv}}\,\chi$.*

Proof (*Sketch*).
(i) Since $\mathscr{F}(\chi)$ is the upper bound of a family of affine functions, then it is a closed and convex function [7].
(ii) Since $\tilde{\chi}(s) \leqslant \chi(s)$, then $ss^* - \chi(s) \leqslant ss^* - \tilde{\chi}(s)$. Hence, $\sup_s\{ss^* - \chi(s)\} \leqslant \sup_s\{ss^* - \tilde{\chi}(s)\}$.
(iii) Since $\mathscr{F}(\chi)(s) = \sup_{s^*}\{\langle s^*, s\rangle - \chi(s^*)\} \geqslant \langle s^*, s\rangle - \chi(s^*)$, then $\chi(s^*) \geqslant \sup_s\{\langle s^*, s\rangle - \mathscr{F}(\chi)(s)\} = \mathscr{F}(\mathscr{F}(\chi))(s^*)$.
(iv) See [7].
(v) $\inf_{s^*}\{\chi(s^*)\} = -\sup_{s^*}\{\langle s^*, 0\rangle - \chi(s^*)\} = -\mathscr{F}(\chi)(0)$ and dually $\inf_{s^*}\mathscr{F}(\chi)(s^*) = -\mathscr{F}(\mathscr{F}(\chi))(0)$ and the inequality comes from (ii).
(vi) Let us suppose that there exists an affine function $\mu(s)$ minorizing $\chi(s)$ and let $\zeta(s)$ be an arbitrary closed convex function such that $\zeta(s) \leqslant \chi(s)$. Let $h(s) = \sup\{\zeta(s), \mathscr{F}(\mathscr{F}(\chi))(s)\}$. This function is closed and convex (since it is the supremum of two closed and convex functions), and $\chi(s) \geqslant h(s) > -\infty$ everywhere. Consequently,

$$h(s) = \mathscr{F}(\mathscr{F}(h))(s) \leqslant \mathscr{F}(\mathscr{F}(\chi))(s) \leqslant h(s)$$

and therefore $h(s) = \mathscr{F}(\mathscr{F}(\chi))(s)$, which implies that $\zeta(s) \leqslant \mathscr{F}(\mathscr{F}(\chi))(s)$, and thus proves (vi). □

A.2. Proof of Theorem 2.2

Consider system (1), with the control (4):

$$\dot{x} = f(x) + g(x)u(x). \tag{18}$$

and let a set-valued map $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ be defined as

$$U(x) = \begin{cases} \left\{ -\dfrac{\gamma(L_g\psi(V(x)))}{|L_g\psi(V(x))|} \cdot \dfrac{L_gV(x)^{\mathrm{T}}}{|L_gV(x)|} \right\} & \text{if } L_gV(x) \neq 0, \\ \sup\limits_{\|v\|=1} \gamma(v) \cdot \mathbb{B} & \text{if } L_gV(x) = 0, \end{cases} \tag{19}$$

where $\mathbb{B}$ is the closed unit ball. $U(x)$ has nonempty, closed and convex values and it is upper semicontinuous. The same holds for the set-valued map

$$F(x) = f(x) + g(x)U(x). \tag{20}$$

Then according to [2, Theorem 7.1], there exist $T \leqslant +\infty$ and an absolutely continuous function $\phi(x,t)$ such that

$$\begin{aligned} &\dot{\phi}(x,t) \in F(\phi(x,t)) \quad \text{for almost all } t \in [0,T), \\ &\phi(x,0) = x. \end{aligned} \tag{21}$$

i.e. (5) is satisfied. To prove strong asymptotic stability, we observe that, for $x \neq 0$,

$$W(x) = \sup_{y\in F(x)} \left\{ \frac{\partial V}{\partial x} y \right\}, \tag{22}$$

$$= \sup_{v\in U(x)} \left\{ \frac{\partial V}{\partial x}(f(x) + g(x)v) \right\}, \tag{23}$$

$$= \begin{cases} L_fV(x) - \dfrac{\gamma(L_g\psi(V(x)))}{\psi'(V(x))} < 0 & \text{if } L_gV(x) \neq 0, \\ L_fV(x) < 0 & \text{if } L_gV(x) = 0. \end{cases} \tag{24}$$

So, since $0 \in F(0)$ and W is continuous, [2, Theorem 15.1] guarantees that the origin is strongly asymptotically stable. □

References

[1] Z. Artstein, Stabilization with relaxed controls, Nonlinear Analysis 7 (1983) 1163–1173.

[2] A.F. Filippov, Differential Equations with Discontinuous Righthand Sides, Kluwer Academic Publishers, Dordrecht, 1988.

[3] R.A. Freeman, P. Kokotović, Robust Nonlinear Control Design: State-Space and Lyapunov Techniques, Birkhäuser, Basel, 1996.

[4] R.A. Freeman, J.A. Primbs, Control Lyapunov functions: new ideas from an old source, Proceedings of the 35th IEEE Conf. on Decision and Control 1996, pp. 3926–3931.

[5] S.T. Glad, On the gain margin of nonlinear and optimal regulators, IEEE Trans. Automat. Control. 29 (1984) 615–620.

[6] B. Hamzi, L. Praly, Ignored input dynamics and a new characterization of control Lyapunov functions, Automatica 36, to appear.

[7] J.-B. Hiriart-Urruty, C. Lemarechal, Convex Analysis and Minimization Algorithms, Springer, Berlin, 1993.
[8] M. Janković, R. Sepulchre, P.V. Kokotović, CLF based designs with robustness to dynamic input uncertainties, Systems Control Lett. 37 (1999) 45–54.
[9] R.E. Kalman, When is linear control system optimal?, Trans. ASME Ser. D. J. Basic Eng. 86 (1964) 1–10.
[10] M. Krstić, Z.-H. Li, Inverse optimal design of input-to-state stabilizing nonlinear controllers, IEEE Trans. Automat. Control 43 (1997) 336–350.
[11] P.J. Moylan, B.D.O. Anderson, Nonlinear regulator theory and an inverse optimal control problem, IEEE Trans. Automat. Control 18 (1973) 460–465.
[12] L. Praly, Lecture Notes on Lyapunov Techniques for the Stabilization of Nonlinear Systems, Université de Paris-Sud, France, 1997.
[13] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, 1970.
[14] R. Sepulchre, M. Jankovic, P.V. Kokotović, Constructive Nonlinear Control, Springer, Berlin, 1997.
[15] E. Sontag, A universal construction of Artstein's theorem on nonlinear stabilization, Systems Control Lett. 13 (1989) 117–123.
[16] A. Teel, L. Praly, On assigning the derivative of disturbance attenuation CLF, Proceedings of the 37th IEEE Conference on Decision and Control, December 1998.
[17] J.N. Tsitsiklis, M. Athans, Guaranteed robustness properties of multivariable nonlinear stochastic optimal regulators, IEEE Trans. Automat. Control 29 (8) (1984) 690–696.