Proceedings of the 42nd IEEE
Conference on Decision and Control
Maui, Hawaii USA, December 2003

Normal Forms for Nonlinear Discrete Time Control Systems

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Keywords: discrete-time, normal forms, homogeneous transformations.

Abstract— **We study the feedback classification of discrete-time control systems whose linear approximation around an equilibrium is controllable. We provide a normal form for systems under investigation.**

I. Introduction

The method of normal forms has been a useful approach in studying the dynamical systems. This method, first introduced by Poincaré in his Ph.D. thesis (see [18]), has been successfully applied by the author to vector fields (differential dynamical systems) and maps (discrete-time systems), in order to provide a change of coordinates in which the system is in a "simplest" form (see also [1]).
For continuous-time control systems with controllable linearization, quadratic normal forms were obtained in [14] using change of coordinates and feedback. This result has been generalized to normal forms of any degree in [13]. Later on, normal forms for control systems with uncontrollable linearization has been derived [12], [16], [20], [22]. Quadratic and cubic normal forms for discrete-time control systems has been treated in [2], [6], [9], [17].
Although this method is formal, it has several applications in control theory. It has been used for the stabilization of systems with uncontrollable linearization, in continuous and discrete-time [4], [7], [5], [8], [10], [16], [17]. It has led to a complete description of symmetries around equilibrium [19], [26], and allowed the characterization of systems equivalent to feedforward forms [23], [24], [25].
In this paper, we propose a normal form, at any degree, for discrete-time control systems whose linearization is controllable.
The paper is organized as following: Section II deals with basic definitions. In Section III, we construct a normal form for discrete-time nonlinear control systems whose linear approximation is controllable. The proofs are given in Section IV.

II. Notations and definitions.

All objects, that is, functions, maps, vector fields, control systems, etc., are considered in a neighborhood of $0 \in \mathbb{R}^n$ and assumed to be C^∞-smooth. For a smooth $\mathbb{R}$-valued function h, defined in a neighborhood of $0 \in \mathbb{R}^n$, we denote by

$$h(x) = h^{[0]}(x) + h^{[1]}(x) + h^{[2]}(x) + \cdots = \sum_{m=0}^{\infty} h^{[m]}(x)$$

its Taylor series expansion at $0 \in \mathbb{R}^n$, where $h^{[m]}(x)$ stands for a homogeneous polynomial of degree m.
Similarly, throughout the paper, for a map ϕ of an open subset of $\mathbb{R}^n$ into $\mathbb{R}^n$ (resp. for a vector field f on an open subset of $\mathbb{R}^n$), we will denote by $\phi^{[m]}$ (resp. by $f^{[m]}$) the term of degree m of its Taylor series expansion at $0 \in \mathbb{R}^n$, that is, each component $\phi_j^{[m]}$ of $\phi^{[m]}$ (resp. $f_j^{[m]}$ of $f^{[m]}$) is a homogeneous polynomial of degree m.
We consider the problem of transforming the discrete-time nonlinear control system

$$\Pi \;:\; x^+ = f(x,u), \quad x(\cdot) \in \mathbb{R}^n \quad u(\cdot) \in \mathbb{R},$$

where $x^+ = x(k+1)$, and $f(x,u) = f(x(k),u(k))$ for any $k \in \mathbb{N}$, by a feedback transformation of the form

$$\Upsilon \;:\; \begin{array}{rcl} z &=& \phi(x) \\ u &=& \gamma(z,v) \end{array}$$

to a simpler form. The transformation Υ brings Π to the system

$$\tilde{\Pi} \;:\; z^+ = \tilde{f}(z,v)\,,$$

whose dynamics are given by

$$\tilde{f}(z,v) = f(\phi^{-1}(z), \gamma(z,v))\,.$$

We suppose that $(0,0) \in \mathbb{R}^n \times \mathbb{R}$ is an equilibrium point, that is, $f(0,0) = 0$, and we denote by

$$\Pi^{[1]} \;:\; x^+ = Fx + Gu\,,$$

its linearization at this point, where

$$F = \frac{\partial f}{\partial x}(0,0), \quad G = \frac{\partial f}{\partial u}(0,0)\,.$$

We will assume that this linearization is controllable, that is

$$\text{span}\left\{ F^i G :\; 0 \le i \le n-1 \right\} = \mathbb{R}^n\,.$$

Let us consider the Taylor series expansion Π^∞ of the system Π, given by

$$\Pi^\infty \;:\; x^+ = Fx + Gu + \sum_{m=2}^{\infty} f^{[m]}(x,u) \qquad (1)$$

and the Taylor series expansion Υ^∞ of the feedback transformation Υ, given by

$$\Upsilon^\infty : \begin{array}{rcl} z & = & \phi(x) = Tx + \sum_{m=2}^{\infty} \phi^{[m]}(x) \\ u & = & \gamma(x,v) = Kx + Lv + \sum_{m=2}^{\infty} \gamma^{[m]}(x,v) . \end{array} \quad (2)$$

Throughout the paper, in particular in formulas (1) and (2), the homogeneity of $f^{[m]}$ and $\gamma^{[m]}$ will be taken with respect to the variables $(x,u)^t$ and $(x,v)^t$ respectively.
We first notice that, because of the controllability assumption, there always exists a linear feedback transformation

$$\Upsilon^1 : \begin{array}{rcl} z & = & Tx \\ u & = & Kx + Lv \end{array}$$

bringing the linear part

$$\Pi^{[1]} : x^+ = Fx + Gu$$

into the Brunovský canonical form (see [11])

$$\Pi^{[1]}_{CF} : z^+ = Az + Bv.$$

Then we study, successively for $m \geq 2$, the action of the homogeneous feedback transformations

$$\Upsilon^m : \begin{array}{rcl} z & = & x + \phi^{[m]}(x) \\ u & = & v + \gamma^{[m]}(x,v) \end{array} \quad (3)$$

on the homogeneous systems

$$\Pi^{[m]} : x^+ = Ax + Bu + f^{[m]}(x,u) . \quad (4)$$

Let us consider another homogeneous system

$$\tilde{\Pi}^{[m]} : z^+ = Az + Bv + \tilde{f}^{[m]}(z,v) . \quad (5)$$

Definition 2.1: We say that the homogeneous system $\Pi^{[m]}$, given by (4), is feedback equivalent to the homogeneous system $\tilde{\Pi}^{[m]}$, given by (5), if there exist a homogeneous feedback transformation Υ^m, of the form (3), which brings the system $\Pi^{[m]}$ into the system $\tilde{\Pi}^{[m]}$ modulo higher order terms.

The starting point is the following proposition giving the equivalence conditions.

Proposition 2.1: The homogeneous feedback transformation Υ^m, defined by *(3)*, brings the homogeneous system $\Pi^{[m]}$, given by *(4)*, into the homogeneous system $\tilde{\Pi}^{[m]}$, given by *(5)*, if and only if the following relation

$$\begin{array}{rcl} \phi_j^{[m]}(Ax+Bu) - \phi_{j+1}^{[m]}(x) & = & \tilde{f}_j^{[m]}(x,u) - f_j^{[m]}(x,u) \\ \phi_n^{[m]}(Ax+Bu) + \gamma^{[m]}(x) & = & \tilde{f}_n^{[m]}(x,u) - f_n^{[m]}(x,u) \end{array}$$

hold for all $1 \leq j \leq n-1$.

III. Main Results.

In this section we will establish our main results. Let us denote the control by $v = z_{n+1}$, and for any $1 \leq i \leq n+1$,

$$\bar{z}_i = (z_1, \cdots, z_i).$$

Our main result for discrete-time nonlinear control systems with controllable linearization is as following.

Theorem 3.1: The control system Π^∞, defined by *(1)*, is feedback equivalent, by a formal feedback transformation Υ^∞ of the form *(2)*, to the normal form

$$\Pi^\infty_{NF} : z^+ = Az + Bv + \sum_{m=2}^{\infty} \bar{f}^{[m]}(z,v) ,$$

where for any $m \geq 2$, we have

$$\bar{f}_j^{[m]}(z,v) = \begin{cases} \sum_{i=j+2}^{n+1} z_1 z_i P_{j,i}^{[m-2]}(\bar{z}_i) & \text{if } 1 \leq j \leq n-1 \\ 0 & \text{if } j = n. \end{cases} \quad (6)$$

As the homogeneous feedback transformations Υ^m leave invariant the terms of degree less than m, Theorem 3.1 follows from a successive application of Theorem 3.2 below.

Theorem 3.2: The homogeneous control system $\Pi^{[m]}$, defined by *(4)*, is feedback equivalent, by a homogeneous feedback transformation Υ^m of the form *(3)*, to the normal form

$$\Pi^{[m]}_{NF} : z^+ = Az + Bv + \bar{f}^{[m]}(z,v) ,$$

where for any $m \geq 2$, the vector field $\bar{f}^{[m]}(z,v)$ is given by *(6)*.

A. Example

Consider the Bressan and Rampazzo pendulum (see [3], [21]) described by the equations

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -g\sin x_3 + x_1 x_4^2 \\ \dot{x}_3 & = & x_4 \\ \dot{x}_4 & = & u, \end{array}$$

where x_1 denotes the length of the pendulum, x_2 its velocity, x_3 the angle of the pendulum with respect to the horizontal, x_4 its angular velocity, and g the gravity constant.

We discretize the system by taking

$$\begin{array}{rclcrcl} \dot{x}_1 & = & x_1^+ - x_1, & & \dot{x}_2 & = & x_2^+ - x_2, \\ \dot{x}_3 & = & x_3^+ - x_3, & & \dot{x}_4 & = & x_4^+ - x_4. \end{array}$$

The system above rewrites

$$\begin{array}{rcl} x_1^+ & = & x_1 + x_2 \\ x_2^+ & = & x_2 - g\sin x_3 + x_1 x_4^2 \\ x_3^+ & = & x_3 + x_4 \\ x_4^+ & = & x_4 + u. \end{array}$$

Let us consider the change of coordinates

$$\begin{array}{rcl} z_1 & = & x_1 \\ z_2 & = & x_2 + x_1 \\ z_3 & = & -g\sin x_3 + 2x_2 + x_1 \\ z_4 & = & -g\sin(x_4 + x_3) + 3x_2 - 2g\sin x_3 + 2x_1 x_4^2 + x_1 \\ v & = & z_4^+ \end{array}$$

whose inverse is such that $x_4 = h(z_1, z_2, z_3, z_4)$ is a smooth function. This change of coordinates takes the system into the form

$$\begin{aligned} z_1^+ &= z_2 \\ z_2^+ &= z_3 + z_1 h^2(z_1, z_2, z_3, z_4) \\ z_3^+ &= z_4 \\ z_4^+ &= v. \end{aligned}$$

Actually the function $h^2(z_1, z_2, z_3, z_4)$ could be decomposed as

$$h^2(z_1, z_2, z_3, z_4) = h_1(z_1, z_2, z_3) + z_4 h_2(z_1, z_2, z_3, z_4)$$

where the 1-jet at 0 of h_1 is zero and $h_2(0) = 0$. Put $H_1(z_1, z_2, z_3) = z_1 h_1(z_1, z_2, z_3)$.

The objective is to show that we can get rid of the terms $H_1(z_1, z_2, z_3)$. Let us suppose that the k-jet at 0 of H_1 is zero.

Consider the change of coordinates $\tilde{z}_1 = z_1, \tilde{z}_2 = z_2, \tilde{z}_3 = z_3 + H_1(z_1, z_2, z_3), \tilde{z}_4 = \tilde{z}_3^+$. This change of coordinates, completed by the feedback $\tilde{z}_4^+ = w$, takes the system into the form

$$\begin{aligned} \tilde{z}_1^+ &= \tilde{z}_2 \\ \tilde{z}_2^+ &= \tilde{z}_3 + \tilde{H}_1(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3) + \tilde{z}_1 \tilde{z}_4 \tilde{H}_2(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4) \\ \tilde{z}_3^+ &= \tilde{z}_4 \\ \tilde{z}_4^+ &= w, \end{aligned}$$

where $\tilde{H}_1(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)$ and $\tilde{H}_2(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3, \tilde{z}_4)$ are some smooth functions. It is enough to remark that the $(k+2)$-jet at 0 of $\tilde{H}_1(\tilde{z}_1, \tilde{z}_2, \tilde{z}_3)$ is zero because the 2-jet of $z_1 z_4 H_2(z)$ is zero. Then by iteration we can cancel terms $H_1(z_1, z_2, z_3)$ and put the system into the desired normal form

$$\begin{aligned} z_1^+ &= z_2 \\ z_2^+ &= z_3 + z_1 z_4 H(z_1, z_2, z_3, z_4) \\ z_3^+ &= z_4 \\ z_4^+ &= v. \end{aligned}$$

IV. Proofs

In this section we will prove our main result. Before let us state the following useful lemma.

Lemma 4.1: If $h^{[m]}(z, u) = h^{[m]}(z_2, \cdots, z_n, u)$ is a homogeneous polynomial depending exclusively on the variables $z_2, \cdots, z_n$ and the control u, then there is a unique homogeneous polynomial $H^{[m]}(z) = H^{[m]}(z_1, \cdots, z_n)$ such that $H^{[m]}(Az + Bu) = h^{[m]}(z, u)$.

The proof of this lemma is straightforward, and hence will be omitted.

A. Proof of Theorem 3.2

The proof will be constructive and based on a inductive argument. Let us consider the system $\Pi^{[m]}$ given by

$$\begin{aligned} x_1^+ &= x_2 + f_1^{[m]}(x, u) \\ &\cdots \\ x_{n-1}^+ &= x_n + f_{n-1}^{[m]}(x, u) \\ x_n^+ &= u + f_n^{[m]}(x, u). \end{aligned} \tag{7}$$

Applying the feedback $v = u + f_n^{[m]}(x, u)$, we can annihilate the terms $f_n^{[m]}(x, u)$, and hence we can assume that $f_n^{[m]}(x, u) = 0$.

Let us suppose that for some $1 \leq j \leq n-1$, the system (7) has been taken to the form

$$\begin{aligned} x_1^+ &= x_2 + f_1^{[m]}(x, u) \\ &\cdots \\ x_j^+ &= x_{j+1} + f_j^{[m]}(x, u) \\ x_{j+1}^+ &= x_{j+2} + \bar{f}_{j+1}^{[m]}(x, u) \\ &\cdots \\ x_{n-1}^+ &= x_n + \bar{f}_{n-1}^{[m]}(x, u) \\ x_n^+ &= u, \end{aligned} \tag{8}$$

where for any $j+1 \leq l \leq n-1$, we have

$$\bar{f}_l^{[m]}(x, u) = \sum_{i=l+2}^{n+1} x_1 x_i P_{l,i}^{[m-2]}(\bar{x}_i).$$

We first decompose the component $f_j^{[m]}(x, u)$ uniquely as follows

$$\begin{aligned} f_j^{[m]}(x, u) &= \sum_{i=j+2}^{n+1} x_1 x_i P_{j,i}^{[m-2]}(\bar{x}_i) \\ &+ \sum_{i=1}^{j+1} x_1 x_i P_{j,i}^{[m-2]}(\bar{x}_i) + R_j^{[m]}(x_2, \cdots, x_n, u). \end{aligned}$$

We consider the feedback transformation

$$\Upsilon^m : \begin{aligned} z &= x + \phi^{[m]}(x) \\ u &= v + \gamma^{[m]}(x, v) \end{aligned}$$

whose components $\phi_1^{[m]}(x), \cdots, \phi_n^{[m]}(x)$, and $\gamma^{[m]}(x, v)$ are defined as following.

Using Lemma 4.1, we define $\phi_j^{[m]}(x)$ such that

$$\phi_j^{[m]}(Ax + Bu) = -R_j^{[m]}(x_2, \cdots, x_n, u) \tag{9}$$

and we take

$$\begin{aligned} \phi_{j+1}^{[m]}(x) &= \sum_{i=1}^{j+1} x_1 x_i P_{j,i}^{[m-2]}(\bar{x}_i) \\ \phi_{j+2}^{[m]}(x) &= \phi_{j+1}^{[m]}(Ax + Bu) \\ &\cdots \\ \phi_n^{[m]}(x) &= \phi_{n-1}^{[m]}(Ax + Bu) \\ \gamma^{[m]}(x) &= \phi_n^{[m]}(Ax + Bu). \end{aligned} \tag{10}$$

The components $\phi_1^{[m]}(x), \cdots, \phi_{j-1}^{[m]}(x)$ could be taken to be zero or arbitraly. Moreover, we can notice that the components $\phi_j^{[m]}(x), \cdots, \phi_n^{[m]}(x)$ did not depend on the control u. Actually, for any $j \leq l \leq n$, we have

$$\phi_l^{[m]}(x) = \phi_l^{[m]}(x_1, \cdots, x_l).$$

Applying Proposition 2.1, we easily deduce that the transformation Υ^m whose components are given by (9)-(10) takes the system (8) into the form

$$\begin{aligned} z_1^+ &= z_2 + f_1^{[m]}(z, u) \\ &\cdots \\ z_{j-1}^+ &= z_j + f_{j-1}^{[m]}(z, u) \\ z_j^+ &= z_{j+1} + \bar{f}_j^{[m]}(z, u) \\ &\cdots \\ z_{n-1}^+ &= z_n + \bar{f}_{n-1}^{[m]}(z, u) \\ z_n^+ &= u, \end{aligned}$$

where for any $j \leq l \leq n-1$, we have

$$\bar{f}_l^{[m]}(z,u) = \sum_{i=l+2}^{n+1} z_1 z_i P_{l,i}^{[m-2]}(\bar{z}_i). \tag{11}$$

This achieves the proof of Theorem 3.2.

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