LECTURE #8

Recall from last time:

There is a \( \mathbb{Q} \)-linear tannakian category \( \text{MTM} = \text{MTM}(\mathbb{Q}) \) of mixed Tate motives, unramified over \( \mathbb{Z} \). Simple objects \( \mathbb{Q}(n), n \in \mathbb{Z} \). Have:

1. \( \text{Ext}^1_{\text{MTM}}(\mathbb{Q}, \mathbb{Q}(2m-1)) = \mathbb{Q}, \ m \geq 1 \)

All other ext groups vanish, except

\[ \text{Ext}^0_{\text{MTM}}(\mathbb{Q}, \mathbb{Q}(0)) = \mathbb{Q} \]

2. \( G_{\text{DR}} := \tau_0 (\text{MTM}, \Omega_{\text{DR}}) \) is an extension

\[ 1 \to K \to G_{\text{DR}} \to G_m \to 1 \]

\[ \overset{\text{pro-unipotent}}{\uparrow} \]

\[ \text{canonical DR splitting} \]

The Lie algebra \( \mathfrak{k} \) of \( K \) is

\[ \mathfrak{k} \cong \mathbb{L}(\mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_7, \mathbb{Z}_9, \ldots)^\wedge \]
where $G_m$ acts on $\mathbb{C}_{2m-1}$ via $x^{-2m+1}$.

**Unipotent path torsors**

$$X = \mathbb{P}^1 - \{0, 1, \infty\} = \text{Spec } \mathbb{Z}[x, x^{-1}, (1-x)^{-1}]$$

Set $$\vec{v}_0 = \frac{\partial}{\partial x} \in T_0 X$$

$$\vec{v}_1 = -\frac{\partial}{\partial x} \in T_1 X$$

![Diagram](image)

For $\alpha, \beta \in \{0, 1\}$, let

$$\alpha \Pi_0 = \pi_0^{un}(X; \vec{v}_\alpha, \vec{v}_0)$$

:= unipotent completion of the paths in $X$ from $\vec{v}_\alpha$ to $\vec{v}_0$

Deligne--Goncharov: This can be
viewed as a (pro-) object of MTM.

\[ \mathcal{O} \left( a \prod_b^{DR} \right) = \bigoplus_{n \geq 0} H^n_{DR}(X) \otimes \omega \]

**Ind. object of MTM**

**Note:** \([\omega_1 | \ldots | \omega_n]\) for \(\omega_1 \otimes \ldots \otimes \omega_n\).

Motivic iterated integrals:

These are certain motivic periods of \(\mathcal{O}(a \prod_b)\), where \(\{a, b\} \in \{v_0, v_1\}\)

\[ I^{\text{mot}}(a; x; b) := [\mathcal{O}(\prod_{b}, a \prod_b; \omega)] \]

Can regard as \(m \{0, 1\}\)

where \(a \prod_b^b\) is the canonical Betti path from \(a\) to \(b\):

\[
\begin{cases}
  o^1 & \text{is} & 0 \rightarrow 1 & a = 0, b = 1 \\
  1^1_0 := o^{1-1} \\
  o^1_0, 1^1_1 \text{ constant loops } \odot \tilde{v}_5, \tilde{v}_1.
\end{cases}
\]


Pause ... this is a little confusing. The following may help:

1) motivic iterated integrals are $\mathcal{O}\left(\mathcal{O}(\mathcal{M}_b)\right)$, matrix entries of $\mathcal{O}(\mathcal{M}_b)$, which is an ind object of $\mathcal{M}_b$. They vanish when $a=b$ and $w = [\omega_{e_1} \ldots \omega_{e_r}]$ with $r=0$.

2) therefore elements of

$$\mathcal{O}_{\mathcal{M}_b} = \mathcal{O}\left(\mathcal{O}(\pi(\mathcal{M}_b; \mathcal{M}_b, \mathcal{M}_b))\right)$$

3) Deligne–Ihara picture:

The loops $\sigma_a \in a\mathcal{T}_a^B$, the path $o_{1}^{B}\sigma_{1}$ and their inverses generate $a\mathcal{T}_b^B$ for $a, b \in \{0, 1\}$. 
(3) Since:
(a) \( B := \{ [\omega_1 | \ldots | \omega_r] : r \geq 0, \epsilon_j \in \{0,1\} \} \)

is a basis of \( \mathcal{O}(0 \mathbb{T}_1^{\mathbb{K}}) \), and

(b) the periods
\[ [\mathcal{O}(0 \mathbb{T}_1), \sigma_{a_j} [\omega_1 | \ldots | \omega_r]] \in \mathcal{O}_{\mathbb{F}} \]

(This can be seen by restricting to \( \mathbb{A}_1^r - \{a_j\} \).)

it follows that \( \mathcal{O}(0 \mathbb{T}_1) \) is generated by \( \mathbb{F} \) and the motivic iterated integrals
\[ I^{\text{mot}}(0; \omega; 1) \]

(4) Similarly
\[ \mathcal{O}[0 \mathbb{T}_0] = \mathcal{O}[0 \mathbb{T}_1] = \mathcal{O}[1 \mathbb{T}_1]. \]

**Notation:** For \( a, b, \epsilon_1, \ldots, \epsilon_r \in \{0,1\} \), set
\[ [a; \epsilon_1, \ldots, \epsilon_r; b] = I^{\text{mot}}(a; \omega; b) \]
where \( \omega = [\omega_1 | \ldots | \omega_r] \).
These form an algebra. The product is the shuffle product.

§ The **regularized period map**

The iterated integral
\[ \int_0^1 [w_1, \ldots, w_r] \]
converges if and only if \( \varepsilon_1 = 0 \) and \( \varepsilon_n = 2 \).

To regularize such divergent integrals we use the fact (from ODE) that

\[ \int_0^{1-s} \left[ [w_1, \ldots, w_n] \right] \in \mathcal{O}^{an}_{\overline{\mathbb{P}}_0} \otimes \mathcal{O}^{an}_{\overline{\mathbb{P}}_1} \left[ \log s, \log t \right] \]

The regularization of \( \int_0^{1-s} [w] \) is obtained by formally setting \( \log s \) and \( \log t \) equal to zero, then evaluating what is left at \( (s, t) = (0, 0) \).

There is a regularized period mapping

\[ \text{per}: \mathcal{O}(\overline{\mathbb{P}}_1) \to \mathbb{C} \]
per : \([a ; \varepsilon_1, \ldots, \varepsilon_v ; b] \rightarrow \int_{a}^{b} \left[ \omega_{\varepsilon_1} | \ldots | \omega_{\varepsilon_v} \right]

**Example:**

(1) \[\int_{0}^{1} \frac{dx}{x} = \text{regularizes to } \int_{t}^{1} \frac{dx}{x} = \text{regularizes to } -\log t = 0.\]

(2) \[\int_{0}^{1} \frac{dx}{x} \frac{dx}{1-x} = -\frac{\pi^2}{6}. \text{ To see this} \]

\[\int_{t}^{1} \frac{dx}{x} \frac{dx}{1-x} = \int_{t}^{1} \frac{dx}{x} \int_{t}^{1} \frac{dx}{1-x} - \int_{t}^{1} \frac{dx}{1-x} \frac{dx}{x} = -\log \left(\frac{1-s}{t}\right) \log \left(\frac{s}{1-t}\right) - \int_{t}^{1} \frac{dx}{1-x} \frac{dx}{x} \text{ regularizes to } 0 \text{ Converges to } S(2) = \frac{\pi^2}{6}.\]
So
\[ \text{per } [0; 0; 1] = 0 \]
\[ \text{per } [0; 0, 1; 1] = -\frac{\pi^2}{6} \]

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\[ \text{Motivic MZVs} \]

Recall that

1. \[ S(n_1, \ldots, n_r) = \sum_{k_1 < \ldots < k_r} \frac{1}{k_1^{n_1} \ldots k_r^{n_r}} \]

where \( n_j \geq 1 \) and \( n_r \geq 2 \),

2. \[ S(n_1, \ldots, n_r) \]

\[ = \int_{0}^{1} \int_{w_1 w_0 \ldots w_0}^{w_1 w_0 \ldots w_0} \ldots \int_{w_1 w_0 \ldots w_0}^{w_1 w_0 \ldots w_0} \]

\[ = (\text{Kontsevich}) \]

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The motivic MZVs are defined by

\[ S_{\text{mot}}(n_1, \ldots, n_r) \]

\[ := [0; \underbrace{1}_{n_1}, \underbrace{0}_{n_2}, \ldots, \ldots, \underbrace{1, 0, \ldots, 0}_{n_r}; 1] \]
They span the ring of motivic \( \text{MZV} \),

\[
\text{MZV}^{\text{mot}} \subset \mathcal{O}(\mathbb{H}).
\]

The image of

\[
\text{per} : \text{MZV}^{\text{mot}} \to \mathbb{C}
\]

is thus the ring of \( \text{MZV} \)s.

One can show that, since

\[
\text{Ext}^1_{\mathcal{O}_M}(\mathbb{Q}, \mathbb{Q}(2n)) = 0 \quad n \geq 1
\]

we have

\[
\Delta \mathcal{S}^{\text{mot}}(2n) = \mathcal{S}^{\text{mot}}(2n) \otimes \mathfrak{x}^{-2n}
\]

This implies that

\[
\mathcal{S}^{\text{mot}}(2n) \subset \mathcal{O}^+ \cdot \mathfrak{p}^{2n}
\]

To compute the multiple, compare their images under \( \text{per} \). Since

\[
\mathcal{S}(2n) = \frac{-B_{2n}}{4n(2n-1)!} \cdot (\pi i)^{2n}
\]

we have

\[
\mathcal{S}^{\text{mot}}(2n) = \frac{-B_{2n}}{4n(2n-1)!} \cdot \mathfrak{p}^{2n}
\]
§ How big is $\MZV^{\text{mot}}$?

Since

$$\MZV^{\text{mot}} \subseteq \mathcal{P}_{\mathcal{O}(T_1)} \subseteq \mathcal{P}^+_{\text{MTM}}$$

we can bound the dimension of $\Gr^w_{2n} \MZV^{\text{mot}}$ by computing $\dim \Gr^w_{2n} \mathcal{P}^+_{\text{MTM}}$ and understanding the relation between $\MZV^{\text{mot}}$ and $\mathcal{P}_{\mathcal{O}(T_1)}$.

To do the first, note that the inclusion

$$\text{MTM}^{ss} \hookrightarrow \text{MTM}$$

induces an inclusion of periods

(1) $\mathcal{P}^+_{\text{MTM}^{ss}} \hookrightarrow \mathcal{P}^+_{\text{MTM}}$

As we noted in an earlier lecture,

$$\mathcal{P}^+_{\text{MTM}^{ss}} = \mathbb{Q}[\Phi].$$

Note that $K$ acts on $\mathcal{P}^+_{\text{MTM}}$ and $\ker\{G_{\text{dr}} \to G_m\}$.
that

\[ \mathfrak{p}_{MTM}^{ss} \subseteq (\mathfrak{g}_{MTM}^{+})^K \]

**Prop:** \( \mathfrak{p}_{MTM}^{+} = (\mathfrak{g}_{MTM}^{+})^K \) \( \Leftarrow \) invariants

**proof.** This follows because an MTM is semi-simple if and only if \( K \) acts trivially on its periods and because the action of \( K \) on periods respects effective periods. \( \square \)

Recall

1. \( G_{DR} \) has a canonical Levi splitting \( G_{DR} \cong G_m \times K_{DR} \).

2. \( \text{TT} (MTM; DR, B) = \text{Spec} \mathfrak{g}_{MTM} \) is a principal \( G_{DR} \) right space.

These imply:
PROP \[ \Phi \ni \rightarrow \text{Spec } \mathcal{C}_{\text{MTM}} \rightarrow \text{Spec } \mathcal{C}_{\text{MTM}}^{\text{ss}} \]

is a principal right \( K_{\text{DR}} \)-bundle.

A section on complex points can be constructed by constructing a ring homomorphism

\[ \Sigma : \mathcal{C}_{\text{MTM}} \otimes \mathbb{C} \rightarrow \mathcal{C}_{\text{MTM}}^{\text{ss}} \otimes \mathbb{C} \]

This can be done as follows:

1. Use \( \mathcal{C}_{\alpha} \subseteq H_{\text{DR}} \) to split the weight filtration of each \( V_{\text{DR}} \)

2. Split the weight filtration of \( V_{\text{B}} \otimes \mathbb{C} \) using the comparison isomorphism

\[ V_{\text{B}} \otimes \mathbb{C} \cong V_{\text{DR}} \otimes \mathbb{C} \]

3. Define \( \Sigma \) by

\[ \Sigma : [v_i, \sigma, w] \mapsto \Sigma [v_i, \tilde{v}_i, \psi_i] \]

where \( \psi = \Xi \sigma \cdot \Xi \tilde{v}_i \in \mathcal{C}_{\tilde{v}_i}^{\text{B}} \otimes \mathbb{C} \)

\( w = \Xi w_i \)

\( w_i \in \mathcal{C}_{\tilde{v}_i}^{\text{B}} \otimes \mathbb{C} \otimes \mathbb{C} \)
Since $\Sigma$ takes effective periods to effective periods, this section extends to a section on effective periods:

**Prop.** The morphism

$$\text{Spec } \mathcal{O}_{\text{MTM}}^+ \to \text{Spec } \mathcal{O}_{\text{MTM}}^{ss} \cong \mathbb{A}_Q^1$$

is a principal $K$-bundle.

$$\Sigma^x \quad \uparrow \quad \downarrow \quad \uparrow \quad \Sigma^x$$

$$\mathbb{A}^1 - \{0\} \quad \hookrightarrow \quad \mathbb{A}^1$$

**Cor.** \( \mathcal{O}_{\text{MTM}}^+ \otimes C \cong C[\ell] \otimes \mathcal{O}(K) \)

The fact, noted above, that

$$\Delta \mathcal{S}^\text{mot}_m = \mathcal{S}^\text{mot}_m \otimes \chi^{-m} \leftrightarrow m \text{ even}$$

implies that

$$\mathcal{O}_{\text{MTM}}^{ss} \cap \mathcal{M} \mathcal{Z} \mathcal{V}^\text{mot} = \mathbb{Q} \left[ \mathcal{S}^\text{mot}_{(2)} \right]$$

$$= \mathbb{Q} \left[ \frac{1}{\ell^2} \right].$$
It follows that

\[
\text{Spec } \mathbb{Z}[\mathbb{V}] \\
\uparrow \qquad \exists
\]

\[
\text{Spec } \mathbb{Q}[\mathbb{V}^2]
\]
is a principal \( \mathbb{K}/\mathbb{N} \) bundle, where

\[
N = \ker \{ \mathbb{K} \to \text{Aut}(\mathbb{Z}[\mathbb{V}]) \}.
\]

**Cor.** \( \mathbb{Z}[\mathbb{V}] \otimes \mathbb{C} \cong \mathbb{Q}[\mathbb{V}^2] \otimes \mathbb{O}(\mathbb{K}/\mathbb{N}). \)

To bound \( \dim \text{Gr}_n \mathbb{Z}[\mathbb{V}] \), we compute

\[
\sum_{n=0}^{\infty} \dim \text{Gr}_n \mathbb{O}(\mathbb{K}) t^n = \frac{1-t^2}{1-t^2-t^3}.
\]

**Proof:**

1. \( \mathbb{K} = \exp(t) \) is pronipotent
2. \( \text{Gr}_n \mathbb{K} \cong \mathbb{L}(z_3, z_5, z_7, \ldots) \)
3. \( \text{Gr}_n \mathbb{K} \cong \mathbb{L}(z_3) \otimes \mathbb{O}(\mathbb{K}) \)
4. \( \text{Gr}_n \mathbb{O}(\mathbb{K}) = \text{graded dual of } \mathbb{L}(\mathbb{K}) \).

where \( V = \mathbb{Q} z_3 \otimes \mathbb{Q} z_5 \otimes \mathbb{Q} z_7 \otimes \ldots \)
(essentially PBW + O(k) = Sym F^k.)

(5) Generating function for $Gr^w V$

is

$$
\beta_v(t) = t^3 + t^5 + t^7 + t^9 + \ldots
$$

$$
= \frac{t^3}{1 - t^2}
$$

So the generating function for $Gr^w V \otimes n$ is

$$
\beta_{v \otimes n}(t) = \beta_v(t)^n.
$$

(6) The generating function for $T(V)$ is

$$
1 + \beta_v(t) + \beta_v(t)^2 + \ldots
$$

$$
= \frac{1}{1 - \beta_v(t)}
$$

$$
= \frac{1}{1 - \frac{t^3}{1 - t^2}}
$$

$$
= \frac{1 - t^2}{1 - t^2 - t^3}.
$$
\[ \sum_{n=0}^{\infty} \dim \text{Gr}_2 \mathfrak{m} \mathfrak{n} \mathcal{F}_{\text{MTM}}^+ t^n = \frac{1+t}{1-t^2-t^3}. \]

**proof.** This follows as the isom \( \mathcal{F}_{\text{MTM}}^+ \otimes C \cong C[\mathfrak{f}] \otimes \mathcal{O}(K) \) respects weight filtrations and as the series for \( C[\mathfrak{f}] \) is
\[ 1 + t + t^2 + \ldots = \frac{1}{1-t}. \]

\[ \sum_{n=0}^{\infty} \dim \text{M2V}_n \leq d_n. \]

**proof:** This follows as the series for \( C[\mathfrak{f}^2] \otimes \mathcal{O}(K/N) \) is bounded by \( \text{M2V}^\text{mot} \otimes C \).
the series for $\mathcal{C}[k^2] \otimes \mathcal{O}(k)$, which is

$$\frac{1}{1-t^2} \cdot \frac{1-t^2}{1-t^2-t^3} \cdot \mathcal{O}$$

Remark: Zagier conjectured equality. Goncharov and Terasoma both deduced this inequality from Levine/Voevodsky

S Brown’s Theorem.

Since $\mathcal{O}(\mathcal{O}_{\mathbb{Q}})$ is an object of $\mathcal{MTM}$, there is an action

$$\mathcal{G}D \rightarrow \text{Aut} \mathcal{O}(\mathcal{O}_{\mathbb{Q}})$$

Theorem (Brown) This action is faithful.

Cor 1: $\mathcal{MTM}$ is generated by $\mathcal{O}(\mathcal{O}_{\mathbb{Q}})$ as a tannakian category
Cor 2: Every period of an object of MTM is an MEV:
\[ P(\mathcal{O}(\mathbb{N}^2)) = \mathcal{S}_{\text{MTM}} \].

Corollary of proof:

**Hoffman's Conjecture**

The MEVs \( \mathcal{S}(n_1, \ldots, n_r) \), where \( n_j \in \{2, 3\} \) span MEV. The corresponding MEVs comprise a \( \mathbb{Q} \) basis of \( \mathcal{S}_{\text{MTM}} \).

**Cor:** MTM is equivalent to the subcategory of \( \mathcal{A} \) generated by \( \mathcal{O}(\mathbb{N}^2) \).