LECTURE #3

Today's subject: Tannaka theory.

§ Motivation: Tannaka duality

- \( k \) a field of char 0

- \( G \in GL_N(k) \) a linear algebraic (and thus) affine group

- \( \mathcal{O}(G) \) its coordinate ring; a Hopf algebra,
  
  . \( \Delta: \mathcal{O}(G) \to \mathcal{O}(G \times G) \cong \mathcal{O}(G) \otimes \mathcal{O}(G) \) (coproduct)

  . (antipode) \( \mathcal{O}(G) \to \mathcal{O}(G) \) dual to mult: \( G \times G \to G \)
  
  . \( \text{Rep}(G) = \) finite dim \( G \)-modules.
The action $G \times V \to V$ corresponds to

\[ V \to V \otimes \mathcal{O}(G) \]

\[ v \mapsto \sum v_j \otimes t_j \]

\[ g \cdot v = \sum t_j(g) v_j. \]

Features of $\text{Rep}(G)$: (abelian category)

- Has tensor product, canonically symmetric and associative;
- Trivial object $1$; canonical isomorphism $V \otimes 1 \cong V \cong 1 \otimes V$;
- $\text{Hom}_G(V, W)$ is a $k$-vector space;
- Duals: $V \cong V^*$ s.t.

\[ \text{Hom}_G(W \otimes V, A) \cong \text{Hom}_G(W, A \otimes V) \]
Matrix entries: For \( v \in V, \varphi \in \mathcal{V} \) have the function

\[
g \mapsto \left\{ k \rightarrow V \twoheadrightarrow V \to k \right\} \in k
\]

Denote it by \([v, V, \varphi]\). It is in \( O(G)\). These form a Hopf algebra: (a) \([v_1, V_1, \varphi_1],[v_2, V_2, \varphi_2]\)

\[
\text{prod} = k \to V_1 \otimes V_2 \xrightarrow{\varphi_1 \otimes \varphi_2} V_1 \otimes V_2 \to k.
\]

\([v_1 \otimes v_2, V_1 \otimes V_2, \varphi_1 \otimes \varphi_2]\]

Coproduct: \( v_1, \ldots, v_d \) basis \( V \)
\( \varphi_1, \ldots, \varphi_d \) dual basis \( \mathcal{V} \)

\[
\text{id} : V \to V = \sum_{i,j} \varphi_i \otimes v_j \cdot v_i \to k \to V
\]

\[
\Delta [v, V, \varphi] = \sum_{i,j} [v, V, \varphi_i] [v_j, V, \varphi_j]
\]
Upshot:

\[ \text{Rep}(G) \rightarrow \mathcal{O}(G) \rightarrow G \]

\[ \uparrow \]

Matrix entries

Fiber Functor:

Have Faithful

\[ w : \text{Rep}(G) \rightarrow \text{Vec}_k. \]

This is a functor of rigid tensor categories. Let

\[ \text{Aut}^\otimes w = \text{group of natural isomorphisms} \]

\[ \eta : w \rightarrow w \]

that preserve \( \otimes \) structure (duals, etc.)

Eq: Each \( g \in G \) — more accurately \( G(k) \) determines an element of \( \text{Aut}^\otimes w \) as
\[
\begin{align*}
V & \xrightarrow{g} V \\
\gamma \downarrow & \quad \downarrow \gamma \\
W & \xrightarrow{g} W
\end{align*}
\]

commutes.

So we have a homomorphism

\[
G \to \text{Aut}^\otimes \Omega
\]

Tannaka duality: This is an isomorphism.

\begin{enumerate}
\item \underline{Neutral Tannakian Categories}
\end{enumerate}

Axiomatize \( \text{Rep}(G) \):

\[ G = \kappa \text{-linear tannakian category.} \]

endowed with (at least one)

faithful functor

\[
\mathfrak{w} : G \to \text{Vec}_K
\]

"fiber functor"
that preserves $\otimes$, duals, distinguished objects. This is a neutral $k$-linear tannakian category. If $K/k$ and $\omega_K: \mathcal{G} \to \text{Vec}_k$ is a fiber functor, then each $\gamma \in \text{Aut}^\otimes \omega_K$ has matrix entries:

$$
\left[ v, \nu, g \right]
$$

where $v: K \to \omega(V)$ and $g: \omega(V) \to K$.

Its value on $\gamma$ is

$$
K \xrightarrow{v} \omega(V) \xrightarrow{\nu} \omega(V) \xrightarrow{g} K
$$

As above, these form a Hopf
algebra over $K$.

$$\tau \_1 \left( \ell, W_k \right) = \text{Spec} \left( \text{this Hopf algebra} \right)$$

The basic result is:

**THM:** If $k$ is a field of char 0 and $\ell$ is a neutral tannakian category with fiber functor $w: \ell \to \text{Vec}_k$

Then $\ell$ is equivalent to

$$\text{Rep} \left( \tau \_1 \left( \ell, w \right) \right).$$

(References on web page.)

**Examples**

1. $\tau \_1 \left( \text{Vec}_k, id \right) = \text{trivial group.}$
(2) The category of graded vector spaces

\[ V = \bigoplus_{n \in \mathbb{Z}} V_n \]

has fundamental group \( \mathbb{G}_{m/k} \).

\[ \begin{array}{ccc}
V & \xrightarrow{t} & V \\
\phi & & \phi \\
W & \xrightarrow{t} & W \\
\end{array} \]

\( t \big/ V_n \) is mult by \( t^n \).

(3) \( \Gamma \) a discrete group

\[ \text{Rep}_k(\Gamma) = \text{finite dimensional} \]

\[ \text{reps of } \Gamma \big/ k. \]

\[ w : \text{Rep}_k(\Gamma) \to \text{Vec}_k \]
Forgets the \( \Gamma \)-action

\[
\pi_1 (\text{Rep}_k (\Gamma), w)
\]

\[
= \varprojlim \ G_{\rho}
\]

where \( \rho \) ranges over the Zariski dense representations

\[
\rho : \Gamma \to G_{\rho} (k).
\]

(4) \( \Gamma \) as in (3).

\( \mathcal{E} = \text{category of unipotent representations of } \Gamma \text{ over } k. \)

\[
\pi_1 (\mathcal{E}, w) = \text{unipotent completion of } \Gamma \text{ over } k.
\]
Variant: Torsors

To neutral tannakian cat $\mathcal{C}$

$\omega_1, \omega_2$: two fiber functors

$\alpha (\mathcal{C}, \omega_1, \omega_2) = \text{Isom} \otimes (\omega_1, \omega_2)$

$\Rightarrow$ affine scheme

principal homog space $/ \pi_1 (\mathcal{C}, \omega_i)$

Holy Grail:

Have $\mathbb{Q}$-linear tannakian cat $M$ of motives over (say) $\mathbb{Z}$. This has fiber functors

$\omega_B : M \to \text{Vec}_\mathbb{Q}$ \hspace{1cm} Betti

$\omega_{DR} : M \to \text{Vec}_\mathbb{Q}$ \hspace{1cm} de Rham

Have affine $\mathbb{Q}$-scheme
\( \overline{t}(M, \omega^g, w^8) \)

\[ \uparrow \]

principal homog space over \( \pi_1(M, w^8) \)

\[ \uparrow \]

"motivic Galois group"

have period map

\[ \mathcal{O}(\pi_1(M, \omega^g, w^8)) \rightarrow \mathbb{C} \]

Gives a canonical \( \mathbb{C} \)-point of \( \overline{t}(M, \omega^g, w^8) \).

\[ [w, V, \Gamma] \mapsto \int_w^V \]
Example:

\[ V = H^n(X, Y) \quad X, Y / \mathcal{Q} \in \mathbb{K} \]

\[ V^{dr} := w_{dr}(V) \]

\[ = H^n_{\mathcal{D}}(X, Y) \quad \in \text{Vec}_\mathcal{Q} \]

\[ V^B = H^n_B(X(\mathcal{C}), \mathcal{C}(\mathcal{C}); \mathbb{Q}) \]

\[ w \in H^0\left(\mathcal{O}^n_X\right), \quad \omega \big|_Y \equiv 0 \]

\[ \mathbb{P} \in H^n_B(X(\mathcal{C}), \mathcal{C}(\mathcal{C}); \mathbb{Q}) \]

Matrix entry:

\[
[w, H^n(X, Y), \mathcal{P}].
\]

Ex: \[ X = \mathcal{G}_m, \quad Y = \{1, x\}. \]

\[ w = \frac{dx}{x}, \quad \mathbb{P} \xrightarrow{1} \int_1^x \]

\[ [\frac{dx}{x}, H^1(\mathcal{G}_m, \{1, x\}), \mathbb{P}] \rightarrow \int_1^x \frac{dx}{x} \sim \log x \]