Periods
Spring 2015

Lecture #1

Landscape of numbers:

\[
\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \overline{\mathbb{Q}} \subset \mathbb{R} = \mathbb{C}
\]

\(\overline{\mathbb{Q}}\) = field of algebraic numbers

Each \(\alpha \in \overline{\mathbb{Q}}\) "contains a finite amount of information"

Encoded in its minimal polynomial

\(m_\alpha(x) \in \mathbb{Q}[x] + \text{relative size}\)

\(\uparrow\)

eg: 3rd largest root of \(m_\alpha(x)\)

"Common wisdom": each transcendental number contains an infinite amount of information as no finite segment
of its decimal expansion determines $\pi$.

However:

\[
\pi = \int \frac{dx}{\sqrt{1-x^2}}
\]

transcendental, yet determined by the inequality $x^2 + y^2 = 1$.

It is an example of a period.

** periods: A period is a complex number (or a sum of complex numbers) whose real and imaginary parts can be expressed as absolutely convergent integrals of the form

\[
\int_{\mathcal{F}} \frac{p(x)}{q(x)} \, dx_1 \ldots dx_n
\]
where

1. $F \subseteq \mathbb{R}^n$ is a region
   \[ f_j(x) \geq 0 \quad j = 1, \ldots, r \]

2. $\rho(x), \eta(x), f_1(x), \ldots, f_r(x) \in \mathbb{Q} [x_1, \ldots, x_n]$

Denote the set of periods by $\mathcal{P}$. It is countable. Fubini’s Theorem implies that $\mathcal{P}$ is a subring of $\mathbb{C}$.

Examples: 

1. $\sqrt{2} \in \mathcal{P}$ as
   \[ \int \frac{dx}{\sqrt{x^2 - 2}} \quad x \geq 0 \]

2. $\pi = \int \int_{x^2 + y^2 \leq 1} dxdy \in \mathcal{P}$
so $\pi^n \in \mathcal{F}$ all $n$.

(3) Every algebraic $a$ is a period:

$$\overline{a} \in \mathcal{F}$$

Enough to show $\overline{\mathbb{Q}} \cap \mathbb{R} \subseteq \mathcal{F}$.

Suppose $a \in \overline{\mathbb{Q}} \cap \mathbb{R}$. Let $m_a(x)$ be its minimal polynomial. Since $m_a(x)$ is separable, $a$ is a simple root of $m_a(x)$.

By replacing $m_a(x)$ by $-m_a(x)$ if necessary we may assume $m'_a(x) > 0$. There are $\beta, \gamma \in \mathbb{Q}$ s.t.
\(1\) \(\beta < \alpha < \gamma\)

\(2\) if \(\alpha < x \leq \beta\), then 
\[m_\alpha(x) \geq 0\]

\(3\) if \(\gamma \leq x < \alpha\), then 
\[m_\alpha(x) < 0\]

Then 
\[\int_{\beta}^{\gamma} dx = \gamma - \alpha \in \mathbb{Q}\]

Since \(\mathbb{Q} \subseteq \mathbb{Q}\), \(\alpha \in \mathbb{Q}\).

\(4\) The logarithm of every non-zero algebraic number is a period:

\[\log_\alpha = \int_{1}^{\alpha} \frac{dx}{x}, \quad \alpha \in \overline{\mathbb{Q}}\]
(5) \( p(x) \in \mathbb{Q}[x] \) cubic, monic square free \((\text{disc} \neq 0)\)

\[ p(x) = (x-\alpha)(x-\beta)(x-\gamma) \in \mathbb{Q}[x]. \]

\[ \int_{\alpha}^{\beta} \frac{dx}{\sqrt{p(x)}} \in \mathbb{R}. \]

\(\text{eg: } \alpha, \beta, \gamma \in \mathbb{R} \quad \alpha < \beta < \gamma.\)

\[ \int_{\alpha}^{\beta} \frac{dx}{\sqrt{p(x)}} = \int \frac{dx}{y} \]

\(d_0 \leq x \leq \beta_0\)

\(p(x) \geq 0\)

(6) Convergent Feynman amplitudes are periods.
Graph trick:
\[ R(x) = t \]
\[ \int \frac{P(x)}{Q(x)} \, dx_1 \ldots dx_n \]
\[ F \]

\[ = \int \int \int R(x) \, 1 \, dt \, dx_1 \ldots dx_n \]
\[ F \quad 0 \]

\[ = \int 1 \, dt \, dx_1 \ldots dx_n \]
\[ \{ (x,t) : x \in \mathbb{R} \}
\[ \{ (x,t) : x \in \mathbb{R} \}
\[ 0 \leq P(x) \leq Q(x) \]
\[ 0 \leq -P(x) \leq -Q(x) \]

Prop: If \( F \subseteq \mathbb{R}^n \) is defined by inequalities
\[ \psi_j(x) \geq 0 \quad \psi_j \in \mathbb{R}[x_1, \ldots, x_n] \]
and if \( \Phi(x) \) is an algebraic function
ie \( \Phi(x) \) is algebraic over \( \mathbb{Q}(x_1, \ldots, x_n) \).
Then
\[ \int \Phi(x) \, dx \ldots dx_n \in \mathcal{P} \]
provided it is absolutely convergent.

Remark: This result (stated in Kontsevich-Zagier) is vague. One issue is that the algebraic function may not be real. I have been unable to find and prove a good formulation of it. Best to move on to the abstract version, which is the most natural for our purposes.
Abstract version:

\( X \) smooth quasi-projective over \( \overline{\mathbb{Q}} \)

\( Y \) closed subvariety defined over \( \overline{\mathbb{Q}} \).

\( \omega \) a closed algebraic \( n \)-form, defined over \( \mathbb{Q} \), s.t. \( \omega \mid Y = 0 \)

\( \mathcal{P} = \text{relative } n \text{-chain in } X(\mathbb{C}) \) (so \( \mathcal{P} \subset Y \))

Then \( \int_{\mathcal{P}} \omega \in \mathbb{Q} \).

Example:

1. \( X = \mathbb{G}_m/\mathbb{Q} \)

\( Y = \{ 1, \alpha \}, \quad \alpha \in \overline{\mathbb{Q}} \)

\( \omega = \frac{dx}{x} \)
\[ \int_{1}^{\infty} \frac{dx}{x} = \log \alpha. \]

2. \[ X : \quad y^2 = p(x) \quad \text{where} \quad p(x) \in \mathbb{Q}[x] \]
   cubic, square free

\[ \omega = \frac{dx}{y} \]

\[ p(x) = (x-\alpha)(x-\beta)(x-\gamma). \]

\[ \int_{\alpha}^{\beta} \frac{dx}{\sqrt{p(x)}} \quad \epsilon \quad \mathfrak{F}. \]

\[ \frac{1}{2} \int_{\alpha}^{\beta} \frac{dx}{y}. \]
CONJECTURAL PICTURE:

Have Galois Theory for

\[
\begin{array}{c}
\mathcal{O} \\
\downarrow \\
\mathbb{Q} \\
\downarrow \\
\mathbb{Q}
\end{array}
\rightarrow
\begin{array}{c}
G \\
\mathcal{O}
\end{array}
\]

preserves all relations between periods

\( G \) affine group over \( \mathbb{Q} \)

(aka profinite group)

\( G \rightarrow G_{\mathbb{Q}} \), will be algebraic

\( \alpha \in \mathcal{O} \), \( G_\alpha = \text{stab of } \alpha \)

= Galois gp of \( \alpha \)

Prop: \( G_\alpha \) infinite \( \Rightarrow \) \( \alpha \) transcendental

Eq: \( G_{\mathbb{Q}} = G_m \). (Tate motives)