

NOTES ON SIMPLICIAL COMPLEXES

Definition: A *simplicial complex* K is a set of finite, non-empty subsets of a set V that satisfies the following properties:

- (i) if $v \in V$, then $\{v\} \in K$;
- (ii) if $\sigma \in K$ and τ is a non-empty subset of σ , then $\tau \in K$.

Elements of V are called the *vertices* of K . If $\{v_0, \dots, v_n\} \in K$, we say that the vertices v_j *span* a simplex of K .

Set

$$K_n = \{\sigma \in K : \#\sigma = n + 1\}.$$

Elements of K_n are called (abstract) *n-simplices*. The subsets of an abstract simplex σ are called the *faces* of σ .

If K is a simplicial complex and $n \in \mathbb{N}$, then

$$\text{sk}_n K := \bigcup_{m \leq n} K_m$$

is a simplicial complex, which is called the *n-skeleton* of K . A simplicial complex K is called an *n-complex* if K_m is empty when $m > n$, or equivalently, $K = \text{sk}_n K$.

An *ordered simplicial complex* is a simplicial complex K together with a partial order on the set V of vertices of K that induces a *total order* on each $\sigma \in K$.

Example: The abstract *n-simplex* $\Delta[n]$ consists of all non-empty elements of the set $[n] := \{0, 1, 2, \dots, n\}$. Its boundary $\partial\Delta[n]$ consists of all non-empty proper subsets of $[n]$. Both are ordered simplicial complexes as the elements of the vertex set $[n]$ are totally ordered. Note that $\partial\Delta[n] = \text{sk}_{n-1} \Delta[n]$.

Example: Every partially ordered set $(P, <)$ gives rise to a simplicial complex $\Delta(P)$, called the *order complex* of P . The set of *n-simplices* of $\Delta(P)$ is the set of all totally ordered $(n + 1)$ -element subsets of P . For example, $\Delta[n]$ is the order complex of $\{0, 1, \dots, n\}$ with its standard order.

Example: Suppose that V is a vector space over a field F and that H_1, \dots, H_n is a finite set of affine hyperplanes. (That is, subsets defined by $L(x) = c$, where $L : V \rightarrow F$ is linear and $c \in F$.) The set P of all finite intersections $H_{j_1} \cap \dots \cap H_{j_k}$ ($k \geq 0$) is partially ordered by inclusion.

Exercise: What is the order complex associated to three lines in \mathbb{R}^2 , that are not concurrent and are pairwise non-parallel?

A Non-Trivial Example of a Simplicial Complex (optional)

Suppose that S is a (smooth) compact oriented surface of genus g (possibly with non-empty boundary). A simple closed curve in S is an imbedded submanifold of S that is diffeomorphic to the circle. Two simple closed curves C_0 and C_1 are *isotopic* if there is a smooth mapping

$$f : S^1 \times [0, 1] \rightarrow S$$

such that

- (i) for each $t \in [0, 1]$, $f_t : S^1 \rightarrow S$ is an imbedding, where $f_t(x) := f(x, t)$, and
- (ii) when $t = 0, 1$, the image of f_t is C_t .

A simple closed curve is *trivial* if it bounds a disk in S or is isotopic to a boundary component.

The *curve complex* of S is the simplicial complex $C(S)$ whose vertices are isotopy classes of non-trivial simple closed curves. The non-trivial simple closed curves C_0, \dots, C_n span an n -simplex of $C(S)$ if the C_j are disjoint and lie in distinct isotopy classes.

We will see later in the course that if S has genus g and r boundary components and if $2g - 2 + r > 0$, then every simplex of $C(S)$ has dimension $\leq 3g - 3 + r$.

Exercise: Show that if S has genus 1 and $r = 0, 1$, then $C(S)$ has vertex set

$$C(S)_0 = \mathbb{Q} \cup \{\infty\} = \mathbb{P}^1(\mathbb{Q})$$

and no simplices of positive dimension. Hint: Show that the simple closed curves in $S^1 \times S^1$ correspond (up to translation) to 1-dimensional subspaces of \mathbb{R}^2 with rational slope.

A theorem of Harer states (among other things) that if $2g - 2 > 0$, then $|C(S)|$ is connected, and if $2g - 2 > 1$, it is simply connected.

Subcomplexes: A *subcomplex* L of a simplicial complex K is a simplicial complex whose set of vertices V_L is a subset of the set V_K of the vertices of K . For each $n \geq 0$, its set L_n of n -simplices is a subset of K_n , the set of n -simplices of K .

Example: The boundary $\partial\Delta[n]$ of the standard n -simplex is a subcomplex of $\Delta[n]$.

Example: The n -skeleton $\text{sk}_n K$ of a simplicial complex K is a subcomplex of K .

Example: Each simplex σ of K can be regarded as a subcomplex with vertex set σ and simplices the set of all subsets τ of σ . The *boundary* $\partial\sigma$ of σ is the subcomplex with the same vertex set, and simplices the *proper* subsets of σ .

Geometric Realization: To each simplicial complex K , one can associate a topological space $|K|$, which is called its *geometric realization*.

For each $\sigma \in K$, let \mathbb{R}^σ be the real vector space with basis the elements of σ . Elements of \mathbb{R}^σ are written

$$(t_v)_{v \in \sigma} = \sum_{v \in \sigma} t_v v$$

where each $t_v \in \mathbb{R}$. The *geometric simplex* corresponding to $\sigma \in K$ is defined to be

$$|\sigma| := \{ \sum_{v \in \sigma} t_v v \in \mathbb{R}^\sigma : t_v \geq 0 \text{ and } \sum_{v \in \sigma} t_v = 1 \}.$$

The coefficients $(t_v)_{v \in V}$ are called the *barycentric coordinates* of $|\sigma|$. When $\sigma = \{v_0, \dots, v_n\}$ we will sometimes denote $|\sigma|$ by $\langle v_0, \dots, v_n \rangle$.

When $\tau \subseteq \sigma \in K$, define $d_\tau^\sigma : |\tau| \hookrightarrow |\sigma|$ to be the natural inclusion that takes $\sum_{v \in \tau} t_v v \in |\tau|$ to $\sum_{v \in \tau} t_v v \in |\sigma|$.

If $\sigma = \{v_0, v_1, \dots, v_n\}$, then the function

$$\Delta^n \rightarrow |\sigma|, \quad (t_0, \dots, t_n) \mapsto \sum_{j=0}^n t_j v_j \in \mathbb{R}^\sigma$$

from the standard n -simplex to $|\sigma|$ is a linear homeomorphism that takes vertices to vertices, edges to edges, etc.

The *geometric realization* $|K|$ of the simplicial complex K is defined to be the quotient

$$|K| = \left(\coprod_{\sigma \in K} |\sigma| \right) / \sim,$$

where \sim is the equivalence relation generated by the relation

$$(x \in |\tau|) \sim (d_\tau^\sigma(x) \in |\sigma|),$$

whenever $\tau \subseteq \sigma$ and $\sigma \in K$.

It is not difficult to show that the actual equivalence relation is

$$(1) \quad (x \in |\sigma|) \sim (y \in |\tau|)$$

if and only if $\sigma \cap \tau \neq \emptyset$ and there exists $z \in |\sigma \cap \tau|$ such that

$$x = d_{\sigma \cap \tau}^\sigma(z) \text{ and } y = d_{\sigma \cap \tau}^\tau(z).$$

Reflexivity and symmetry are clear; the proof of transitivity requires a little work.¹

The geometric realization of the abstract n -simplex $\Delta[n]$ is the standard n -simplex Δ^n .

The *interior* $\text{int } |\sigma|$ of the geometric simplex $|\sigma|$ is defined by

$$\text{int } |\sigma| = \{\sum_{v \in \sigma} t_v v \in \mathbb{R}^\sigma : t_v > 0 \text{ and } \sum_{v \in \sigma} t_v = 1\}.$$

Note that if σ is a 0-simplex, then $\text{int } |\sigma| = |\sigma|$ and that (as sets)

$$(2) \quad |\sigma| = \coprod_{\tau \subseteq \sigma} \text{int } |\tau| \text{ and } |\partial \sigma| = \coprod_{\tau \subset \sigma} \text{int } |\tau|$$

The important properties of the geometric realization are:

- (i) As a set (but not as a topological space)

$$|K| = \coprod_{\sigma \in K} \text{int } |\sigma|.$$

This follows directly from (1), which implies quite directly that a point in the interior of one simplex cannot be identified with a point in the interior of another. The fact that every point of $|K|$ lies in the interior of some simplex follows from (2) and the surjectivity of the quotient map $\coprod_{\sigma} |\sigma| \rightarrow |K|$.

- (ii) For each $\sigma \in K$, the composite $q \circ j_\sigma$:

$$|\sigma| \xrightarrow{j_\sigma} \coprod_{\sigma \in K} |\sigma| \xrightarrow{q} |K|$$

is injective and is the inclusion of a subspace. (So we will view each geometric simplex $|\sigma|$ as a subspace of $|K|$.) Injectivity follows directly from (i) and (2). That j_σ is a subspace inclusion is proved Appendix B.

- (iii) A subset of $|K|$ is open (resp. closed) if and only if its intersection with each geometric simplex $|\sigma|$ of K is open (resp. closed) in $|\sigma|$. This follows from the fact that $\coprod |\sigma| \rightarrow |K|$ is a quotient mapping and the fact that each j_σ is a subspace inclusion.
- (iv) If L is a subcomplex of K , then there is a natural continuous mapping $|L| \rightarrow |K|$ which is injective and is the inclusion of a subspace. This follows from the first three properties (i), (ii) and (iii).

¹You have to show that if $x \in |\sigma|$, $y \in |\tau|$ and $z \in |\rho|$, and if $x \sim y$ and $y \sim z$, then $\sigma \cap \tau \cap \rho$ is non-empty and there exists $w \in |\sigma \cap \tau \cap \rho|$ such that $d_{\sigma \cap \tau \cap \rho}^\sigma(w) = x$, $d_{\sigma \cap \tau \cap \rho}^\tau(w) = y$, and $d_{\sigma \cap \tau \cap \rho}^\rho(w) = z$.

- (v) A closed subset C of $|K|$ is compact if and only if it lies in the union of a finite number of geometric simplices:

$$C \subseteq \bigcup_{j=1}^n |\sigma_j|.$$

Alternatively, C is compact if and only if $C \cap \text{int } |\sigma|$ is non-empty for only finitely many $\sigma \in K$. These two statements are easily seen to be equivalent. The second statement is proved in the Appendix C.

- (vi) Every simplicial complex is locally contractible. That is, each point of $|K|$ has a base of contractible neighbourhoods. This is proved in Appendix A. In fact, we will prove that $|K|$ is locally conical.

Exercise: Suppose that K is a simplicial complex with a vertex v_o such that if $\sigma \in K$, then $\sigma \cup \{v_o\} \in K$. Let

$$L = \{\sigma \in K : v_o \notin \sigma\}.$$

This is a sub-complex of K . Show that $|K|$ is the cone $C(|L|)$ over $|L|$ with cone point $v_o \in |K|$. That is, show that the mapping

$$|L| \times [0, 1] \rightarrow |K|$$

defined by $(x, t) \mapsto tx + (1 - t)v_o$ induces a homeomorphism

$$C(|L|) := (|L| \times [0, 1]) / (|L| \times 0) \xrightarrow{\cong} |K|$$

Example: If P is a partially ordered set with an element x_o such that $x_o \leq x$ for each $x \in P$, then the order complex of P is contractible. More precisely, it is the cone over the order complex of the partially ordered set $P - \{x_o\}$.

Triangulation:

A *triangulation* of a topological space X is simplicial complex K and a homeomorphism $\phi : |K| \rightarrow X$.

Example: Since Δ^n is homeomorphic to the n -ball B^n , the boundary $\partial\Delta^n$ is a triangulation of S^{n-1} . This triangulation can be made explicit by considering S^{n-1} to be the sphere in the affine hyperplane $t_0 + t_1 + \cdots + t_n = 1$ of radius $\sqrt{n/(n+1)}$ centered at the barycenter $(e_0 + \cdots + e_n)/(n+1)$ of Δ^n . This is the unique $(n-1)$ -sphere in \mathbb{R}^{n+1} that contains each vertex of Δ^n . The triangulation of this S^{n-1} is obtained by radially projecting $\partial\Delta^n$ onto the sphere from the center of the sphere. \square

Two important results:

- (i) (S. Cairns (1934)) Every smooth manifold admits a smooth triangulation. (That is, the inclusion of each $|\sigma|$ into the manifold is a smooth imbedding.) If N is a closed submanifold of a smooth manifold M , it is possible to triangulate M so that N is a subcomplex. Note that not every topological manifold is triangulable.
- (ii) (S. Lojasiewicz (1964), H. Hironaka (1975)) Every real semi-algebraic set (i.e., a set defined by a finite number of real polynomial equations and inequalities) admits a real semi-algebraic triangulation. Even stronger, if X is a real semi-algebraic set and X_1, \dots, X_n is a finite collection of real algebraic subsets of X , each closed in X , then there is a triangulation of X such that each X_j is a subcomplex. Since real and complex algebraic varieties are real algebraic sets, it follows that every real and complex algebraic variety admits a semi-algebraic triangulation and that closed subvarieties can be realized as subcomplexes. Lojasiewicz's result is stronger in that he proves that every real semi-analytic set is triangulable.²

Simplicial Maps: Suppose that K and L are simplicial complexes with vertex sets V_K and V_L , respectively. A *simplicial map* $f : K \rightarrow L$ is a function $f : V_K \rightarrow V_L$ with the property that if $\sigma \in K$, then $f(\sigma) \in L$. (Here $f(\sigma)$ is the image of the subset σ of V_K under f .) Note that $\#f(\sigma) \leq \#\sigma$.

A simplicial mapping $f : K \rightarrow L$ induces a map $|f| : |K| \rightarrow |L|$ of geometric realizations. It is induced by the mapping

$$\coprod_{\sigma \in K} |\sigma| \rightarrow \coprod_{\tau \in L} |\tau| \rightarrow |L|$$

where the restriction of the first mapping to $|\sigma|$ is the unique affine linear map

$$f_\sigma : |\sigma| \rightarrow |f(\sigma)|$$

that takes the vertex v of σ to the vertex $f(v)$ of $|f(\sigma)|$. This is given by the formula:

$$f_\sigma\left(\sum_{v \in \sigma} t_v v\right) = \sum_{v \in \sigma} t_v f(v) \in |f(\sigma)|.$$

²Hironaka's paper — *Triangulations of algebraic sets*, Algebraic geometry, Proc. Sympos. Pure Math., 29 (1975) — is quite accessible and gives a more direct approach in the semi-algebraic case than Lojasiewicz's paper.

This is easily seen to respect the equivalence relation that defines $|K|$, and therefore induces a continuous mapping $|f| : |K| \rightarrow |L|$.

Subdivision:

Often, for technical reasons, it will be necessary to subdivide a simplicial complex — that is, divide each of its simplices into smaller pieces. One standard way (but not the only way) to do this is by barycentric subdivision.

The *barycenter* of the geometric simplex $\langle v_0, \dots, v_n \rangle$ in the real vector space V is its “center of mass”:

$$\frac{1}{n+1} \sum_{j=0}^n v_j.$$

Figure 1 show the barycenter of a 2-simplex. We shall denote the

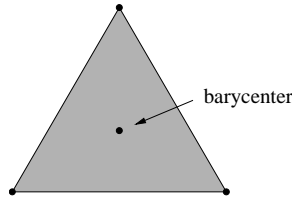


FIGURE 1. Barycenter of a 2-simplex

barycenter of a geometric simplex $|\sigma|$ by $\hat{\sigma}$.

The *barycentric subdivision* $\text{sd } K$ of a simplicial complex K is the (abstract) simplicial complex whose vertices consist of all simplices of K (ordered by inclusion) and whose n -simplices consist of all chains

$$\sigma_0 \subset \sigma_1 \subset \dots \subset \sigma_n$$

where each $\sigma_j \in K$ and σ_{j-1} is a proper face of σ_j .

Inclusion of simplices defines a partial order on the vertices of $\text{sd } K$ with the property that the vertices of each simplex of $\text{sd } K$ are totally ordered. Thus $\text{sd } K$ is an ordered simplicial complex.

Figure 2 shows the barycentric subdivision of a 2-simplex whose vertex set is $\{0, 1, 2\}$. Here, the barycenter of the simplex $\langle j_0, \dots, j_k \rangle$ is denoted $j_0 j_1 \dots j_k$. The edge $j \subset jk$ of the barycentric subdivision is denoted by $[j, jk]$ and the 2-simplex $j \subset jk \subset jk\ell$ of the barycentric subdivision by is denoted by $[j, jk, jk\ell]$. The orientations of the edges are indicated with arrows. Note that if σ is a permutation of $\{0, 1, 2\}$, then the orientation of the 2-simplex $[\sigma(0), \sigma(01), \sigma(012)]$ is $\text{sgn}(\sigma)$ times the orientation of the 2-simplex $\langle 0, 1, 2 \rangle$.

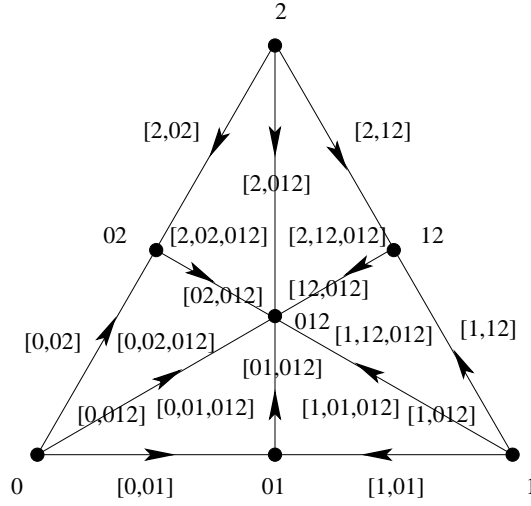


FIGURE 2. Barycentric subdivision of a 2-simplex

If σ is the standard n -simplex $\langle 0, 1, \dots, n \rangle$, then the n -simplices of $\text{sd } \sigma$ are

$$[\sigma(0), \sigma(01), \sigma(012), \dots, \sigma(012 \dots n)],$$

where σ is a permutation of $\{0, 1, \dots, n\}$. Note that these are in 1-1 correspondence with the orderings of the vertices of σ . There are thus $(n+1)!$ n -simplices in $\text{sd } \sigma$. The vertices of $[0, 01, 012, \dots, 012 \dots n]$ have barycentric coordinates

$$(1, 0, \dots, 0), (1, 1, 0, \dots, 0)/2, \dots, (\overbrace{1, \dots, 1}^k, 0, \dots, 0)/k, \dots, (1, 1, \dots, 1)/(n+1).$$

The affine span of these points in $|\sigma|$ is exactly the n -simplex

$$\{(t_0, t_1, \dots, t_n) \in |\sigma| : t_0 \geq t_1 \geq t_2 \geq \dots \geq t_n\}.$$

By applying $\sigma \in S_{n+1}$ to this, we see that the points

$$\sigma(0), \sigma(01), \dots, \sigma(012 \dots n)$$

span the simplex

$$\{(t_0, t_1, \dots, t_n) \in |\sigma| : t_{\sigma(0)} \geq t_{\sigma(1)} \geq t_{\sigma(2)} \geq \dots \geq t_{\sigma(n)}\}.$$

From this it follows that $|\sigma|$ is the union of the geometric simplices of $\text{sd } \sigma$. These intersect along lower dimensional simplices of $\text{sd } \sigma$.

For an n -simplex σ , define the continuous mapping

$$\phi_\sigma : |\text{sd } \sigma| \rightarrow |\sigma|$$

to be the unique affine linear mapping that takes the vertex τ of $\text{sd}(\sigma)$ to its barycenter $\hat{\tau} \in |\sigma|$. This is a homeomorphism.

If $\sigma \in K$, then $\text{sd } \sigma$ is a subcomplex of $\text{sd } K$. Denote the induced map on realizations by $p_\sigma : |\text{sd } \sigma| \hookrightarrow |\text{sd } K|$. It is a subspace inclusion. There is a unique continuous mapping $\phi : |\text{sd } K| \rightarrow |K|$ such that the diagram

$$\begin{array}{ccc} |\text{sd } \sigma| & \xrightarrow{\phi_\sigma} & |\sigma| \\ p_\sigma \downarrow & & \downarrow j_\sigma \\ |\text{sd } K| & \xrightarrow{\phi} & |K| \end{array}$$

commutes. It is a homeomorphism.

The Star Covering:

Simplicial complexes have a natural, combinatorially defined, open covering. It is a useful technical tool.

Suppose that K is a simplicial complex and that $\sigma \in K$.

Define

$$\text{st}(\sigma) := \bigcup_{\tau \supseteq \sigma} \text{int } |\tau|.$$

This is open in $|K|$. To see this, note that $|\tau| \cap \text{st}(\sigma)$ is non empty if and only if $\tau \supseteq \sigma$. (This is because the $\text{int } |\tau|$ are disjoint in $|K|$.) It follows that $|\tau| \cap \text{st}(\sigma)$ is open in $|\tau|$ for all $\tau \in K$.

Figure 3 illustrates this: v and w are adjacent vertices, $\text{st}(v)$ is the interior of the left-hand shaded hexagon and $\text{st}(w)$ is the interior of the right-hand shaded hexagon. Their intersection, $\text{st}\langle v, w \rangle$, is the dark shaded area.

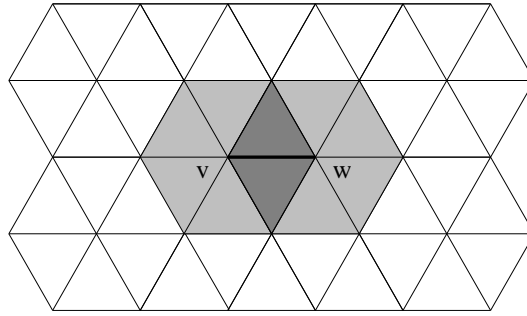


FIGURE 3. $\text{st}\langle v, w \rangle = \text{st}(v) \cap \text{st}(w)$

Since a simplex τ contains σ if and only if it contains each vertex of σ , it follows that

$$\text{st}(\sigma) = \bigcap_{v \in \sigma} \text{st}(v).$$

The *star covering* of $|K|$ is the open covering

$$\{\text{st}(v) : v \text{ is a vertex of } K\}.$$

Simplicial Approximation:

Continuous mappings between geometric realizations of simplicial complexes are homotopic (after subdivision) to simplicial maps. This is a powerful technical result. More precisely, suppose that K_1 and K_2 are simplicial complexes and that L is a (possibly empty) subcomplex of K_1 . Suppose that $\phi_L : L \rightarrow K_2$ is a simplicial map.

Theorem 1. *If K_1 is finite and if $f : |K_1| \rightarrow |K_2|$ is a continuous mapping whose restriction to $|L|$ is $|\phi|$, then there is a positive integer m and a simplicial map $\phi : \text{sd}^m K_1 \rightarrow K_2$ whose restriction to L is $\text{sd}^m \phi_L$ such that f is homotopic to $|\phi|$ rel $|L|$.*

Sketch of proof when L is empty. Since $|K_1|$ is compact, the open covering $\{f^{-1}\text{st}(w) : w \text{ is a vertex of } K_2\}$ has a Lebesgue number³ $\delta > 0$. Subdivide K_1 until the diameter of the star $\text{st}(v)$ of each vertex v of $\text{sd}^m K_1$, is $< \delta$.⁴ It follows that for each vertex v of $\text{sd}^m K_1$, $f(\text{st}(v))$ is contained in $\text{st}(w)$ for some vertex w of K_2 . For each v choose such a w — call it $\phi(v)$. This defines a simplicial map $\phi : \text{sd}^m K_1 \rightarrow K_2$. Indeed, if v_0, \dots, v_n span a simplex of $\text{sd}^m K_1$, then $\cap_j \text{st}(v_j)$ is non-empty. Since

$$f^{-1}\left(\bigcap_{j=0}^n \text{st}(\phi(v_j))\right) \supseteq \bigcap_{j=0}^n \text{st}(v_j) \neq \emptyset$$

so that $\cap_j \text{st}(\phi(v_j))$ is non-empty and $\phi(v_0), \dots, \phi(v_n)$ span a simplex of K_2 . Observe that if σ is a simplex of $\text{sd}^m K_1$ and $x \in |\sigma|$, then $f(x)$ lies in $\text{st}(\phi(\sigma))$. This allows us to define a homotopy from f to $|\phi|$ by $F(x, t) = tf(x) + (1 - t)\phi(x)$. \square

This argument is easily modified to include the case when L is non-empty.

The assumption that K_1 be finite can be dropped if we consider subdivisions more general than barycentric subdivisions. (See Spanier, for example.)

³Every open covering \mathcal{U} of a compact space X has a Lebesgue number $\delta > 0$. That is a positive number such that every subset of X of diameter $< \delta$ is contained in some $U \in \mathcal{U}$.

⁴This is possible. If K is a simplicial complex of dimension d , then the diameter (with respect to the original metric on $|K|$) of the star cover of $|\text{sd}^m K|$ is bounded by $2(d/(d+1))^{m/2}$, which goes to 0 as $m \rightarrow \infty$.

Example: If $n > m > 0$, then every continuous mapping $f : (S^m, e_0) \rightarrow (S^n, e_0)$ is homotopic to a constant mapping rel e_0 . This follows from the Simplicial Approximation Theorem above as follows. Triangulate S^m and S^n so that e_0 is a vertex of each. (For example, S^r is homeomorphic to $\partial\Delta^{r+1}$.) By the theorem, there is a $k > 0$ such that f is homotopic rel e_0 to the realization of a simplicial mapping $\phi : \text{sd}^k \partial\Delta[m+1] \rightarrow \partial\Delta[n+1]$. But such a mapping is not surjective as $m < n$. Since S^n minus a point is homeomorphic to \mathbb{R}^n , it follows that f is nullhomotopic. This is a non-trivial result. It says, for example, that $\pi_m(S^n, e_0) = 0$ when $m < n$.

An Application:

The topology of every complex algebraic variety (not necessarily smooth, not necessarily compact) is finite. More precisely, every complex algebraic variety has the homotopy type of the geometric realization of a finite simplicial complex. In particular, the fundamental group of every complex algebraic variety is finitely presented and all of its homology (and cohomology) groups are finitely generated.

To see this, we use the fact that every complex algebraic variety has a triangulation. An algebraic variety Z is said to be *quasi-projective* if $Z = X - Y$, where X is a projective variety and Y is a closed subvariety. If Y is non-empty, Z is not compact, so it is not clear that Z has the homotopy type of a finite simplicial complex. However, as previously remarked, there is a simplicial complex K (necessarily finite as X is compact), a subcomplex L , and a triangulation

$$\phi : (|K|, |L|) \rightarrow (X, Y).$$

Thus Z is homeomorphic to $|K| - |L|$. That this is homotopy equivalent to a finite complex follows from the following exercise.

Exercise: Suppose that K is a simplicial complex and that L is a subcomplex. We may assume that no simplex of K has more than one face that is a simplex of L . (If necessary, replace (K, L) by $(\text{sd } K, \text{sd } L)$.) Set

$$U = \bigcup \{\text{int } |\sigma| : |\sigma| \cap |L| \text{ is non-empty}\}.$$

Show that

- (i) $|L|$ a deformation retraction of U ;
- (ii) $|K| - U$ is a deformation retraction of $|K| - |L|$.

Deduce that if K is a finite simplicial complex, then $|K| - |L|$ has the homotopy type of a finite simplicial complex.

APPENDIX A. SIMPLICIAL COMPLEXES ARE LOCALLY CONICAL

Suppose that K is a simplicial complex and that $\sigma \in K$. Note that $|\sigma|$ has a natural metric induced from the euclidean metric via the inclusion $|\sigma| \rightarrow \mathbb{R}^\sigma$:

$$\text{dist}_\sigma((t_v)_{v \in \sigma}, (s_v)_{v \in \sigma}) = \left(\sum_{v \in \sigma} (t_v - s_v)^2 \right)^{1/2}.$$

Note that if $\tau \subset \sigma$, the inclusion $d_\tau^\sigma : |\tau| \rightarrow |\sigma|$ is an isometry (i.e., distance preserving).

For $r > 0$ and $x \in |\sigma|$, set

$$B_\sigma(x, r) = \{y \in |\sigma| : \text{dist}_\sigma(x, y) < r\}$$

and

$$S_\sigma(x, r) = \{y \in |\sigma| : \text{dist}_\sigma(x, y) = r\}.$$

Suppose that $x \in |K|$. Then there is a unique $\sigma \in K$ such that $x \in \text{int } |\sigma|$. If $\dim \sigma > 0$, choose $\epsilon > 0$ such that $B_\sigma(x, \epsilon) \subset \text{int } |\sigma|$. If σ is a vertex, take $\epsilon < \sqrt{2}/2$. Define

$$N(x, \epsilon) = \bigcup_{\tau \supseteq \sigma} B_\tau(x, \epsilon).$$

Since $N(x, \epsilon) \cap |\rho| \neq \emptyset$ implies that $\rho \supseteq \sigma$, and since (by the choice of ϵ), $N(x, \epsilon) \cap |\tau| = B_\tau(x, \epsilon)$ is open in $|\tau|$ when $\tau \supseteq \sigma$, $N(x, \epsilon)$ is open in $|K|$.

The union

$$\partial N(x, \epsilon) := \bigcup_{\tau \supseteq \sigma} S_\tau(x, \epsilon)$$

is closed in $|K|$. The mapping

$$(\partial N(x, \epsilon) \times [0, 1]) / (\partial N(x, \epsilon) \times 0), \quad (y, t) \mapsto (1 - t)x + ty$$

induces a homeomorphism $\text{cone}(\partial N(x, \epsilon)) \rightarrow N(x, \epsilon)$.

It follows that each point $x \in |K|$ has a base $\{N(x, r) : r < \epsilon\}$ of conical (and therefore contractible) neighbourhoods.

APPENDIX B. $j_\sigma : |\sigma| \hookrightarrow |K|$ IS A SUBSPACE INCLUSION

Suppose that K is a simplicial complex, that $\sigma \in K$, and that U is an open subset of $|\sigma|$. We have to construct an open subset V of $|K|$ whose intersection with $|\sigma|$ is U .

Here is one construction of V — there is another, which uses cones over barycenters. For each $x \in U$, there is $\epsilon(x) > 0$ such that

$$B_\sigma(x, \epsilon(x)) \subseteq U.$$

Define

$$V = \bigcup_{x \in U} N(x, \epsilon(x)).$$

This is a union of open subsets of $|K|$, and therefore open in $|K|$. Moreover,

$$V \cap |\sigma| = \bigcup_{x \in U} B_\sigma(x, \epsilon(x)) = U$$

as required.

APPENDIX C. CHARACTERIZATION OF COMPACT SUBSETS OF $|K|$

Suppose that C is a closed subset of $|K|$. Consider the following 3 statements:

- (i) $C \cap \text{int } |\sigma|$ is non-empty for only finitely many $\sigma \in K$;
- (ii) there is a finite subcomplex L of K such that $C \subseteq |L|$;
- (iii) C is compact.

We will prove these three statements are equivalent. The first implies the second. Just take

$$L = \{\tau \in K : \tau \subseteq \sigma \text{ and } C \cap \text{int } |\sigma| \neq \emptyset\}.$$

The second statement implies the third as $|L|$, being the quotient of the compact space $\coprod_{\tau \in L} |\tau|$, is compact.

To complete the proof, we show that the third implies the first. Suppose that C is compact. Let

$$S = \{\sigma \in K : C \cap \text{int } |\sigma| \neq \emptyset\}.$$

If S is finite, then we are done. If not, choose for each $\sigma \in S$, choose a point $x_\sigma \in C \cap \text{int } |\sigma|$. set $T := \{x_\sigma : \sigma \in S\}$.

Since S is infinite, the compactness of C implies that S has a point of accumulation $x_o \in C$, say. This lies in $\text{int } |\sigma_o|$ for some $\sigma_o \in K$. By replacing x_{σ_o} by x_o if necessary, we may assume that $x_o \in S$. There is a sequence x_{σ_n} in $T - \{x_o\}$ that converges to x_o . Since $\text{dist}(x_o, |\partial \sigma_o|) > 0$, we may assume that σ_o is a face of each σ_n . For convenience, set $x_n = x_{\sigma_n}$.

We show that there is a continuous function $f : |K| \rightarrow \mathbb{R}$ such that

$$f(x_o) = 1 \text{ and } f(x_n) = 0 \text{ when } n > 0.$$

The existence of such a function implies that x_o is not a limit point of the x_σ . This is a contradiction, so S must be finite.

To construct f , we need only construct a family of functions $f_\sigma : |\sigma| \rightarrow \mathbb{R}$ such that

- (i) f_σ vanishes on $\{x_n\} \cap |\sigma|$;
- (ii) $f_{\sigma_o}(x_o) = 1$;
- (iii) if $\tau \subset \sigma$, then $f_\sigma|_{|\tau|} = f_\tau$.

Set $d = \dim \sigma_o$. Start by defining f to be identically 1 on $|\text{sd}_d K|$. We construct f_σ when $\sigma \in K_n$ and $n > d$ by induction on n . Suppose that $n > d$ and that f_τ has already been defined for all lower dimensional simplices. Choose a point $z_o \in \text{int } |\tau|$. Choose this to be x_n if $\tau = \sigma_n$ and any point otherwise. Observe that

$$C(|\partial\tau|) \rightarrow |\tau|, \quad (x, t) \mapsto tx + (1 - t)z_o$$

is a homeomorphism from the closed cone over $|\partial\tau|$ to $|\tau|$. Define f_τ by

$$f_\tau(x, t) = tf_{\partial\tau}(x).$$

This agrees with f on the boundary of $|\tau|$ and takes the value 0 at z_o , as required.

Remark: A straightforward elaboration of this argument can be used to show that $|K|$ is a normal topological space. (See Spanier for details.)