Richard Hain February 16, 2023

Math 612
Problem Set 3

Due: Thursday March 9, 2023

1. In this problem $R = \mathbb{Z}$. Compute $\operatorname{Ext}(\mathbb{Z}/N, \mathbb{Z})$ and

$$\operatorname{Tor}(\mathbb{Z}/p^m, \mathbb{Z}/q^n) \text{ and } \operatorname{Ext}(\mathbb{Z}/p^m, \mathbb{Z}/q^n).$$

Here p and q are prime numbers (not necessarily distinct) and n, m are positive integers.

2. Show that if R is a PID and if each $H_k(X; R)$ is a finitely generated R-module, then

$$\operatorname{rank} H_k(X; R) = \operatorname{rank} H^k(X; R).$$

Deduce that if X is a finite cell complex, then

$$\sum_{k=0}^{\infty}(-1)^k \operatorname{rank}_R H^k(X; R) = \chi(X) := \sum_{k=0}^{\infty}(-1)^k \#(k\text{-cells of } X).$$

3. Show that $\operatorname{Ext}_R^\bullet$ and $\operatorname{Tor}_\bullet^R$ respect finite direct sums. That is, there are natural isomorphisms

$$\operatorname{Tor}_n^R(\oplus_j A_j, \oplus_k B_k) \cong \bigoplus_{j,k} \operatorname{Tor}_n^R(A_j, B_k)$$

and

$$\operatorname{Ext}_R^n(\oplus_j A_j, \oplus_k B_k) \cong \bigoplus_{j,k} \operatorname{Ext}_R^n(A_j, B_k).$$

4. Show that if P is a projective R-module, then

$$\operatorname{Ext}_R^n(P, M) = 0 \text{ and } \operatorname{Tor}_n^R(P, M) = \operatorname{Tor}_n^R(M, P) = 0$$

for all R-modules M and all $n > 0$.

4. Show that if G is a finite group of order d and M is a *trivial* G-module, then d annihilates $H_k(G; M)$ and $H^k(G; M)$ for all $k > 0$. (This result holds for all G-modules.)

5. Suppose that d is a positive integer. Show that for all topologial pairs (X, A) there are long exact sequences

$$\cdots \longrightarrow H_{j+1}(X, A; \mathbb{Z}/d) \xrightarrow{\beta} H_j(X, A; \mathbb{Z}) \xrightarrow{\times d} H_j(X, A; \mathbb{Z})$$
$$\xrightarrow{r} H_j(X, A; \mathbb{Z}/d) \xrightarrow{\beta} H_{j-1}(X, A; \mathbb{Z}) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow H^{j-1}(X,A;\mathbb{Z}/d) \xrightarrow{\beta} H^j(X,A;\mathbb{Z}) \xrightarrow{\times d} H^j(X,A;\mathbb{Z})$$
$$\xrightarrow{r} H^j(X,A;\mathbb{Z}/d) \xrightarrow{\beta} H^{j+1}(X,A;\mathbb{Z}) \longrightarrow \cdots$$

that are natural in (X,A), where in both cases r is induced by the reduction mod d map $\mathbb{Z} \to \mathbb{Z}/d$. These sequences are *Bockstein sequences* and the connecting homomorphism β is called a *Bockstein map*.

Compute the homology and cohomology Bockstein sequences with $d = 2$ and 3 for $\mathbb{RP}^4$.

Hint: first show that the sequence

$$0 \longrightarrow C_\bullet(X,A;\mathbb{Z}) \xrightarrow{\times d} C_\bullet(X,A;\mathbb{Z}) \xrightarrow{\text{mod } d} C_\bullet(X,A;\mathbb{Z}/d) \longrightarrow 0$$

is exact.