Richard Hain February 16, 2023

Math 612
Problem Set 2

Due: Tuesday Februarty 28, 2023.

1. Denote the n-torus $(S^1)^n$ by T_n. Compute the action of the automorphism

$$\sigma : T_n \to T_n, \quad \sigma : x \mapsto -x$$

on $H_\bullet(T_n; R)$ and $H^\bullet(T_n; R)$ for all coefficient rings R.

2. Suppose that $\pi : Y \to X$ is a covering map of finite degree d. Fix a base ring R.

(i) Show that each singular simplex $\sigma : \Delta^n \to X$ has d lifts $\sigma_j : \Delta^n \to Y$, $j = 1, \dots, d$. (That is, $\pi \circ \sigma_j = \sigma$, each j.)

(ii) Define an R-module map $\pi^* : C_\bullet(X) \to C_\bullet(Y)$ by

$$\pi^*(\sigma) = \sigma_1 + \cdots + \sigma_d = \sum_{\substack{\tau:\Delta^n \to Y_\tau \\ \pi\circ\tau=\sigma}} \tau$$

Show that π^* is a chain map and that $\pi_* \circ \pi^*$ is multiplication by d.

(iii) Deduce that for all R-modules M, there are maps

$$\pi^* : H_\bullet(X; M) \to H_\bullet(Y; M) \text{ and } \pi_* : H^\bullet(Y; M) \to H^\bullet(X; M)$$

such that the composites

$$H_\bullet(X; M) \xrightarrow{\pi^*} H_\bullet(Y; M) \xrightarrow{\pi_*} H_\bullet(X; M)$$

$$H^\bullet(X; M) \xrightarrow{\pi^*} H^\bullet(Y; M) \xrightarrow{\pi_*} H^\bullet(X; M)$$

are multiplication by d. Deduce that if $H^j(Y; M) = 0$ (resp. $H_j(Y; M) = 0$), then d annihilates $H^j(X; M)$ (resp. $H_j(X; M)$).

(iv) Deduce that if $\pi : S^n \to X$ is a d-fold covering map, then $H_j(X; \mathbb{Z})$ is annhilated by d when $0 < j < n$.

(v) Regard S^{2m-1} as the unit sphere in $\mathbb{C}^n$. Show that the group $\boldsymbol{\mu}_d$ of dth roots of unity acts fixed point freely on S^{2m-1} by multiplication. Deduce that $H_j(S^{2m-1}/\boldsymbol{\mu}_d; \mathbb{Z})$ is annhilated by d when $0 < j < 2m - 1$. (We'll see that these groups are cyclic.)

3. Assume now that the covering in the previous problem is Galois with Galois group G. (This is finite of order d.) Suppose that $d \in R^\times$.

(i) Show that if V is an $R[G]$-module, then there is a unique G-submodule V' of V such that

$$V = V^G \oplus V'$$

where $V^G = \{v \in V : gv = v \text{ all } g \in G\}$. Hint: Take $V' = \{v \in V : \sum_{g \in G} gv = 0\}$.

(ii) Show that the singular chain *complex* of Y decomposes

$$C_\bullet(Y) = C_\bullet(Y)^G \oplus C_\bullet(Y)'$$

(Here, coefficients R are understood.)

(iii) Show that the restriction of π_* to $C_\bullet(Y)^G$ is an isomorphism with inverse $d^{-1}\pi^*$.

(iv) Deduce that for all R-modules M,

$$\pi^* : H^\bullet(X; M) \to H^\bullet(Y; M)^G \text{ and } \pi_* : H_\bullet(Y; M)^G \to H_\bullet(X; M)$$

are isomorphisms.