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MATH 612
PROBLEM SET 1

Due: Tuesday, February 9, 2023

Fix a (commutative) ground ring R and an R-module M.

1. Show that if $(X, A) = \coprod_{\alpha \in A}(X_\alpha, A_\alpha)$, then there are natural isomorphisms

$$H^\bullet(X, A; M) \cong \prod_{\alpha \in A} H^\bullet(X_\alpha, A_\alpha; M) \text{ and } H_\bullet(X, A; M) \cong \bigoplus_{\alpha \in A} H_\bullet(X_\alpha, A_\alpha; M)$$

2. Suppose that $\{(X_\alpha, x_\alpha) : \alpha \in A\}$ is a collection of pointed spaces. Suppose that in each X_α, there is a neighbourhood U_α of x_α that deformation retracts onto $\{x_\alpha\}$. Show that the inclusions $i_\alpha : X_\alpha \to \bigvee X_\alpha$ induces a natural isomorphism

$$\prod i_\alpha^* : \widetilde{H}^\bullet(\bigvee_{\alpha \in A} X_\alpha; M) \to \prod_{\alpha \in A} \widetilde{H}^\bullet(X_\alpha; M)$$

3. Suppose that Y is a compact orientable surface of genus g. Suppose that $D_1, \dots, D_n$ are $n > 0$ disjoint closed disks embedded in Y. Let

$$X = Y - \bigcup_{j=1}^{n} \mathring{D}_j.$$

That is, X is a genus g surface with n boundary components, each of which is a circle. Compute

$$H^\bullet(X; R) \text{ and } H^\bullet(X, \partial X; M)$$

where ∂X denotes the boundary of X, which is the disjoint union of n circles.

4. Let X be a compact orientable surface of type described in the previous in the previous problem. Let Z be the surface obtained by attaching n Möbius bands to X, one to each of its boundary components. Compute the integral and $\mathbb{F}_2$ cohomology of Z.