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Math 611
Worksheet 2

1. The goal of this problem is to show that $\mathrm{GL}_n(\mathbb{R})$ has two connected components which are distinguished by the sign of the determinant.

(i) Briefly explain why $\mathrm{O}(n)$ is a deformation retraction of $\mathrm{GL}_n(\mathbb{R})$. (No need to give a complete proof. Deduce that the inclusion $\mathrm{O}(n) \hookrightarrow \mathrm{GL}_n(\mathbb{R})$ induces a bijection on connected components.
(ii) Show that, when $n > 1$, $\mathrm{SO}(n)$ acts transitively on S^{n-1} and that the stabilizer of $e_n \in S^{n-1}$ is $\mathrm{SO}(n-1)$.
(iii) Show that the mapping $\mathrm{SO}(n)/\mathrm{SO}(n-1) \to S^{n-1}$ that takes $A \cdot \mathrm{SO}(n-1)$ to Ae_n is a homeomorphism.
(iv) Show that if $n \geq 2$, then $\mathrm{SO}(n)$ is connected if and only if $\mathrm{SO}(n-1)$ is connected. Deduce that $\mathrm{SO}(n)$ is connected for all $n \geq 1$ and that $\mathrm{O}(n)$ has two connected components.
(v) Show that $\mathrm{GL}_n(\mathbb{R})$ has two connected components:

$$\{A \in \mathrm{GL}_n(\mathbb{R}) : \det A > 0\} \text{ and } \{A \in \mathrm{GL}_n(\mathbb{R}) : \det A < 0\}.$$

2. Suppose that V is a real vector space of dimension d, where $d > 0$. Two ordered bases $(v_1, \dots, v_d)$ and $(w_1, \dots, w_d)$ of V determine an element A of $GL_d(\mathbb{R})$ via:

$$(v_1, \dots, v_d) = (w_1, \dots, w_d)A.$$

Define two ordered bases to be equivalent if the matrix A that relates them has positive determinant. An *orientation* of V is an equivalence class of ordered bases of V. Denote the equivalence class of the ordered basis $(v_1, \dots, v_d)$ by $[v_1, \dots, v_d]$.

(i) Show that each V has exactly two orientations.
(ii) Show that if σ is a permutation of $\{1, \dots, d\}$, then

$$[v_{\sigma(1)}, \dots, v_{\sigma(d)}] = \mathrm{sgn}(\sigma)[v_1, \dots, v_d].$$

(The orientation opposite to $[v_1, \dots, v_d]$ is denoted by $-[v_1, \dots, v_d]$.)

5. (Orientations of simplices) Suppose that V is a finite dimensional vector space and that $v_0, \dots, v_n$ are affine independent elements of V. Denote the simplex $\langle v_0, \dots, v_n\rangle$ they span by Δ. Show that the vector space $T\Delta$ of vectors tangent to Δ is

$$T\Delta = \left\{\textstyle\sum_{j=0}^n a_j v_j : a_j \in \mathbb{R}, \sum_{j=0}^n a_j = 0\right\}.$$

By definition, an orientation of the simplex Δ is an orientation of the vector space $T\Delta$. Each ordering of the vertices of Δ determines an orientation of Δ as follows: the orientation determined by the vertex ordering $v_0 < v_1 < \cdots < v_n$ is defined to be

$$[v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0].$$

Denote it by $\langle v_0, \ldots, v_n \rangle$.[1] Show that

(i) If $w_0 := v_0 + \cdots + v_n$ is non-zero, then $w_1, \ldots, w_n$ is a positive basis of $T\Delta$ if and only if $w_0, w_1, \ldots, w_n$ is a positive basis of V.

(ii) if $1 \leq j \leq n$, then

$$[v_0 - v_j, v_1 - v_j, \ldots, v_{j-1} - v_j, v_{j+1} - v_j, \ldots, v_n - v_j] \\ = (-1)^j [v_1 - v_0, \ldots, v_n - v_0].$$

(iii) if σ is a permutation of $\{0, 1, \ldots, n\}$, then

$$\langle v_{\sigma(0)}, v_{\sigma(1)}, \ldots, v_{\sigma(n)} \rangle = \operatorname{sgn}(\sigma) \langle v_0, \ldots, v_n \rangle.$$

That is, each ordering of the vertices of a simplex orient the simplex; two orderings determine the same orientation if and only if they differ by an even permutation. This should help explain why we care about and need ordered simplicial complexes.

3. The goal of this problem is to show that the orientation induced on the jth face Δ_j of $\langle v_0, \ldots, v_n \rangle$ is $(-1)^j \langle v_0, \ldots, \widehat{v_j}, \ldots, v_n \rangle$. We will use the notation of the previous problem.

(i) Show that the vector

$$w_j := -v_j + \frac{1}{n+1} \sum_{j=0}^{n} v_j \in T\Delta$$

is an outward normal to the jth face of Δ.[2]

(ii) The standard convention for orienting a boundaries ("outward normal first") says that a basis $u_1, \ldots, u_{n-1}$ of the tangent space $T\Delta_j$ of the boundary is positively oriented when

$$w_j, u_1, \ldots, u_{n-1}$$

is a positively oriented basis of $T\Delta$. (Nothing to prove here!)

[1] I admit that this notation is mildly ambiguous. But we will see that it is a convenient abuse of notation. More accurately, when we write $\langle v_0, \ldots, v_n \rangle$ we will mean the oriented simplex spanned by the ordered set of vectors $v_0, v_1, \ldots, v_n$.

[2] Here, take $v_0, v_1, \ldots, v_n$ to be an orthonormal basis of $\mathbb{R}^{\{v_0, \ldots, v_n\}}$.

(iii) Show that the orientation $\langle v_0, \ldots, v_n \rangle$ of Δ induces the orientation $\langle v_1, \ldots, v_n \rangle$ on the 0th face Δ_0.

(iv) Use action of the symmetric group of $\{0, 1, \ldots, n\}$ and the results of the previous problem (or otherwise) to deduce that the the orientation induced on Δ_j by $\langle v_0, \ldots, v_n \rangle$ is

$$(-1)^j \langle v_0, \ldots, \widehat{v_j}, \ldots, v_n \rangle.$$