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Math 611
Worksheet 1

One goal of this worksheet is to show that the braid group B_3 is isomorphic (naturally) to the fundamental group of the complement of the trefoil knot in S^3 and to use this isomorphism to derive the standard presentation of B_3. A second goal is to construct a homomorphism $B_3 \to \mathrm{SL}_2(\mathbb{Z})$.

1. A knot is a smooth embedding of the circle S^1 in S^3. Suppose that m and n are relatively prime integers. Regard S^3 as the unit sphere in $\mathbb{C}^2$. Let K be the knot $S^1 \to S^3$ by $w \mapsto (c_1 w^n, c_2 w^m)$, where c_1, c_2 are positive real numbers satisfying $c_1^2 + c_2^2 = 1$ and $c_1^m = c_2^n$. This is a *torus knot of type* (n, m).[1]

(i) Use van Kampen's Theorem to compute a presentation of $\pi_1(S^3 - K, x_o)$. (Hint: use results from Problem Set 2. You will need to choose your base point to lie on $T(c_1^2) - K$. Your generators should be the "spines" of the two solid tori. That is, apart from a brief excursion to and from the base point, they can be chosen to lie on the coordinate axes.

(ii) Compute $\pi_1\big(\mathbb{C}^2 - \{(x,y) : x^m = y^n\}, *\big)$.

2. As in class, let

$$Y_n = \mathbb{C}^n - \bigcup_{j<k} \Delta_{j,k},$$

where $\Delta_{j,k}$ is the hyperplane $\lambda_j = \lambda_k$. Let $X_n = \Sigma_n \backslash Y_n$, where the symmetric group Σ_n acts on Y_n by permuting the coordinates. As in class, this is regarded as the space of monic polynomials of degree n with distinct roots. Denote the discriminant of the polynomial

$$T^n + a_1 T^{n-1} + \cdots + a_{n-1} T + a_n$$

by $D(a_1, \ldots, a_n)$. The map $X_n \to \mathbb{C}^n - D^{-1}(0)$ defined by

$$(\lambda_1, \ldots, \lambda_n) \mapsto \big(-\sigma_1(\lambda), \sigma_2(\lambda), \ldots, (-1)^n \sigma_n(\lambda)\big).$$

Observe that $X_n = \mathbb{C}^n - D^{-1}(0)$. Set

$$Y_n^o = Y^n \cap \{\lambda \in \mathbb{C}^n : \lambda_1 + \cdots + \lambda_n = 0\}.$$

and

$$X_n^o = \Sigma_n \backslash Y_n^o = X_n \cap \{T^n + a_1 T^{n-1} + \cdots + a_{n-1} T + a_n : a_1 = 0\}.$$

[1]The *trefoil knot* is the torus knot of type (2,3) or (3,2).

(i) Show that the map $\mathbb{C} \times Y_n^o \to Y_n$ defined by

$$(\overline{\lambda}, (\lambda_1, \dots, \lambda_n)) \mapsto (\lambda_1 + \overline{\lambda}, \dots, \lambda_n + \overline{\lambda})$$

is a homeomorphism.[2]

(ii) Deduce that X_n is homeomorphic to $\mathbb{C} \times X_n^o$ and that $B_n = \pi_1(X_n^o, *)$.

(iii) The discriminant of the cubic $T^3 - aT - b$ is $4a^3 - 27b^2$. Deduce that

$$X_3^o = \mathbb{C}^2 - \{(a,b) : 4a^3 = 27b^2\}$$

and that B_3 is isomorphic to the fundamental group of the complement of the trefoil knot.

(iv) What braids correspond to letting (a,b) move about a unit circle in the a-axis? In the b-axis? What base points are you using? How do you join them up?

3. Abstract group theory (with soul): the goal of this problem is to use the presentation of the complement of the trefoil knot to derive the standard presentation of B_3. Let

$$G_1 = \langle u, s : u^3 = s^2 \rangle \text{ and } G_2 = \langle \sigma_1, \sigma_2 : \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

(i) Show that the homomorphism $\langle u, s \rangle \to G_2$ defined by

$$u \mapsto \sigma_2\sigma_1 \text{ and } s \mapsto \sigma_2\sigma_1\sigma_2$$

induces a homomorphism $G_1 \to G_2$.

(ii) Show that the homomorphism $\langle \sigma_1, \sigma_2 \rangle \to G_1$ defined by

$$\sigma_1 \mapsto s^{-1}u^2 \text{ and } \sigma_2 \mapsto u^{-1}s$$

induces a homomorphism $G_2 \to G_1$.

(iii) Show that these homomorphisms are inverses and hence isomorphisms.

(iv) Show that $u^3 = s^2$ is central in G_1, so that the corresponding element $(\sigma_2\sigma_1)^3$ is central in B_3. Which braid is this?

4. The goal of this problem is to show that these groups map to $\mathrm{SL}_2(\mathbb{Z})$ and to give a hint as to why B_3 is isomorphic to

$$\widetilde{\mathrm{SL}_2(\mathbb{Z})} := \text{inverse image of } \mathrm{SL}_2(\mathbb{Z}) \text{ in } \widetilde{\mathrm{SL}_2(\mathbb{R})}$$

which is a central extension

$$0 \to \mathbb{Z} \to \widetilde{\mathrm{SL}_2(\mathbb{Z})} \to \mathrm{SL}_2(\mathbb{Z}) \to 1.$$

[2] It is a diffeomorphism, biholomorphism, ...

Consider the following elements of $\mathrm{SL}_2(\mathbb{Z})$:

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \text{and } U = ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

(i) Show that S has order 4, U has order 6 and that $S^2 = U^3 = -I$.
(ii) Show that there is a homomorphism $\varphi : G_1 \to \mathrm{SL}_2(\mathbb{Z})$ that takes u to U and s to S. Show that it takes the central element of the previous problem to $-I$.
(iii) Deduce that there is a homomorphism $B_3 \to \mathrm{SL}_2(\mathbb{Z})$.

It is true that

$$\mathrm{SL}_2(\mathbb{Z}) = \langle S, U, Z : S^2 = U^3 = Z, Z^2 = 1\rangle$$

and that

$$\widetilde{\mathrm{SL}_2(\mathbb{Z})} = \langle S, U : S^2 = U^3\rangle,$$

so that B_3 is isomorphic to $\widetilde{\mathrm{SL}_2(\mathbb{Z})}$. You can find a proof of the first statement in the handout on $\mathrm{SL}_2(\mathbb{Z})$ on the class web page. The only proof of the second statement that I can think of at the moment, although elementary, requires some knowledge of the theory of elliptic curves.[3]

[3]An elliptic curve is a compact Riemann surface isomorphic to the quotient $\mathbb{C}/\Lambda$ of $\mathbb{C}$ by a lattice. Equivalently, it cubic curve in $\mathbb{CP}^2$ with equation of the form $y^2 = x^3 - ax - b$.