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Math 261
Algebraic Topology I
Topology Summary

Subspace Topology: Suppose that X is a topological space and that A is a subset. The *subspace topology* on A has open sets

$$\{U \cap A : U \text{ is open in } X\}.$$

With this topology on A, the inclusion $j : A \hookrightarrow X$ is continuous. It is characterized by the following property: a function $f : Z \to A$ from a topological space to A is continuous (w.r.t. the subspace topology on A) if and only $f \circ j : Z \to X$ is continuous:

$$\begin{tikzcd} A \arrow[r, "j"] & X \\ Z \arrow[u, "f"] \arrow[ur, "j\circ f"'] & \end{tikzcd}$$

Quotient Topology: Suppose that X is a topological space and that $q : X \to Y$ is a surjective function. The *quotient topology* on Y has open sets

$$\{U \subseteq Y : q^{-1}(U) \text{ is open in } X\}.$$

With this topology the quotient mapping $q : X \to Y$ is continuous. The quotient topology on Y induced by q is characterized by the following property: a function $f : Y \to Z$ from Y to a topological space Z is continuous if and only if $f \circ q : X \to Z$ is continuous:

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr, "f\circ q"'] & Y \arrow[d, "f"] \\ & Z \end{tikzcd}$$

A continuous function $q : X \to Y$ is called a *quotient mapping* if and only if it surjective and Y has the quotient topology.

Example: One common special case is where Γ is a group (not necessarily discrete) that acts continuously on a topological space X. The quotient $\Gamma\backslash X$ of X by Γ is the set of Γ-orbits endowed with the quotient topology. A useful fact is that the quotient mapping $q : X \to \Gamma\backslash X$ is open. To see this note that if U is an open subset of X, then

$$q^{-1}q(U) = \bigcup_{g\in\Gamma} gU$$

which is open in X as each gU is open in X. This implies that $q(U)$ is open in $\Gamma\backslash X$. $\square$

The quotient and subspace topologies are related by the following easily proved result: Suppose that we have the commutative diagram

$$\begin{tikzcd} A \arrow[r, "j"] \arrow[d, "p"'] & X \arrow[d, "q"] \\ B \arrow[r, "i"] & Y \end{tikzcd}$$

of functions, where $A = q^{-1}(B)$, B, X, and Y are topological spaces, j is a subspace inclusion, q a quotient mapping, i is injective and p surjective. If q is open, then i is a subspace inclusion if and only if p is a quotient mapping.

Example: Consider the case where $A = \mathbb{R}^{m+1}$, $X = \mathbb{R}^{n+1} - \{0\}$, $B = \mathbb{RP}^m$, and $Y = \mathbb{RP}^n$, where $m \leq n$ and A is imbedded linearly in X. Since $\mathbb{RP}^n$ is the quotient of X by a group action, the projection q is open. It follows that $\mathbb{RP}^m$ is a subspace of $\mathbb{RP}^n$. $\square$

The condition that $A = q^{-1}(B)$ is necessary in the previous result. For example, consider the diagram

$$\begin{tikzcd} \mathbb{R} \arrow[r, "j"] \arrow[d, "p"'] & \mathbb{R}^2 \arrow[d, "q"] \\ \mathbb{R} \arrow[r, "i"] & \mathbb{R}^2/\mathbb{Z}^2 \end{tikzcd}$$

where $j(x) = (x, ax)$ and $a \in \mathbb{R}$ is a fixed irrational number. In this case, p is the identity. If we make $i : \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$ the inclusion of a subspace, then p is not a quotient mapping. If we make $p : \mathbb{R} \to \mathbb{R}$ into a quotient mapping, the i is not the inclusion of a subspace.

Coproduct Topology: Suppose that $\{X_\alpha : \alpha \in A\}$ is a family of topological spaces indexed by the set A. The disjoint union (aka, the *coproduct*) of the X_α is denoted by

$$\coprod_{\alpha \in A} X_\alpha.$$

The coproduct topology on $\coprod_\alpha X_\alpha$ has open sets

$$\{\textstyle\coprod_\alpha U_\alpha : U_\alpha \subset X_\alpha \text{ is open in } X_\alpha\}$$

With this topology, each of the inclusions $i_\alpha : X_\alpha \to \coprod X_\alpha$ is the inclusion of a subspace. The coproduct topology is characterized by the following property: a function

$$f : \coprod_{\alpha} X_{\alpha \in A} \to Z$$

to a topological space Z is continuous if and only if the restriction $f \circ i_\alpha$ of f to each X_α is continuous:

$$\begin{tikzcd}
\coprod_\alpha X_\alpha \arrow[r, "f"] & Z \\
X_\alpha \arrow[u, "i_\alpha"] \arrow[ur, "f \circ i_\alpha"'] &
\end{tikzcd}$$

If $\{U_\alpha : \alpha \in A\}$ is an open covering of A, then the mapping

$$\coprod_{\alpha \in A} U_\alpha \to X$$

induced by the inclusions $U_\alpha \hookrightarrow X$ is a quotient mapping.

If $X = F_1 \cup F_2 \cup \cdots \cup F_n$, where each F_j is a closed subset of X, then the mapping

$$F_1 \amalg F_2 \amalg \cdots \amalg F_n \to X$$

is a quotient mapping. (It is important that n be finite here.) This statement is equivalent to the *glueing lemma.*

Product Topology: Suppose that $\{X_\alpha : \alpha \in A\}$ is a family of topological spaces indexed by the set A. The product topology on $\prod_\alpha X_\alpha$ has basis the open sets:

$$\{\textstyle\prod_\alpha U_\alpha : U_\alpha \subset X_\alpha \text{ is open in } X_\alpha \text{ and } X_\alpha = U_\alpha \text{ for almost all } \alpha\},$$

where "almost all" means "all but a finite number." Each of the projections $p_\alpha : \prod_\alpha X_\alpha \to X_\alpha$ is a quotient mapping. The product topology is characterized by the following property: a function

$$f : Z \to \prod_{\alpha} X_{\alpha \in A}$$

from a topological space Z is continuous if and only if the αth component $p_\alpha \circ f : Z \to X_\alpha$ is continuous:

$$\begin{tikzcd}
Z \arrow[r, "f"] \arrow[dr, "p_\alpha \circ f"'] & \prod_\alpha X_\alpha \arrow[d, "p_\alpha"] \\
& Z
\end{tikzcd}$$

A common notation for f is $f = (f_\alpha)$, where $f_\alpha := p_\alpha \circ f$.

Basic Facts from Topology:

(i) A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.
(ii) A compact subspace of a Hausdorff space is closed.
(iii) The continuous image of a compact space is compact.