Richard Hain

## Math 261 Algebraic Topology I Topology Summary

SUBSPACE TOPOLOGY: Suppose that X is a topological space and that A is a subset. The subspace topology on A has open sets

 $\{U \cap A : U \text{ is open in } X\}.$ 

With this topology on A, the inclusion  $j : A \hookrightarrow X$  is continuous. It is characterized by the following property: a function  $f : Z \to A$  from a topological space to A is continuous (w.r.t. the subspace topology on A) if and only  $f \circ j : Z \to X$  is continuous:



QUOTIENT TOPOLOGY: Suppose that X is a topological space and that  $q: X \to Y$  is a surjective function. The *quotient topology* on Y has open sets

 $\{U \subseteq Y : q^{-1}(U) \text{ is open in } X\}.$ 

With this topology the quotient mapping  $q: X \to Y$  is continuous. The quotient topology on Y induced by q is characterized by the following property: a function  $f: Y \to Z$  from Y to a topological space Z is continuous if and only if  $f \circ q: X \to Z$  is continuous:

$$X \xrightarrow{q} Y$$

$$f \circ q \qquad \downarrow^{f}$$

$$Z$$

A continuous function  $q: X \to Y$  is called a *quotient mapping* if and only if it surjective and Y has the quotient topology.

**Example:** One common special case is where  $\Gamma$  is a group (not necessarily discrete) that acts continuously on a topological space X. The quotient  $\Gamma \setminus X$  of X by  $\Gamma$  is the set of  $\Gamma$ -orbits endowed with the quotient topology. A useful fact is that the quotient mapping  $q: X \to \Gamma \setminus X$  is open. To see this note that if U is an open subset of X, then

$$q^{-1}q(U) = \bigcup_{g \in \Gamma} gU$$

which is open in X as each gU is open in X. This implies that q(U) is open in  $\Gamma \setminus X$ .

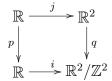
The quotient and subspace topologies are related by the following easily proved result: Suppose that we have the commutative diagram



of functions, where  $A = q^{-1}(B)$ , B, X, and Y are topological spaces, j is a subspace inclusion, q a quotient mapping, i is injective and p surjective. If q is open, then i is a subspace inclusion if and only if p is a quotient mapping.

**Example:** Consider the case where  $A = \mathbb{R}^{m+1}$ ,  $X = \mathbb{R}^{n+1} - \{0\}$ ,  $B = \mathbb{R}\mathbb{P}^m$ , and  $Y = \mathbb{R}\mathbb{P}^n$ , where  $m \leq n$  and A is imbedded linearly in X. Since  $\mathbb{R}\mathbb{P}^n$  is the quotient of X by a group action, the projection q is open. It follows that  $\mathbb{R}\mathbb{P}^m$  is a subspace of  $\mathbb{R}\mathbb{P}^n$ .

The condition that  $A = q^{-1}(B)$  is necessary in the previous result. For example, consider the diagram



where j(x) = (x, ax) and  $a \in \mathbb{R}$  is a fixed irrational number. In this case, p is the identity. If we make  $i : \mathbb{R} \to \mathbb{R}^2/\mathbb{Z}^2$  the inclusion of a subspace, then p is not a quotient mapping. If we make  $p : \mathbb{R} \to \mathbb{R}$  into a quotient mapping, the i is not the inclusion of a subspace.

COPRODUCT TOPOLOGY: Suppose that  $\{X_{\alpha} : \alpha \in A\}$  is a family of topological spaces indexed by the set A. The disjoint union (aka, the *coproduct*) of the  $X_{\alpha}$  is denoted by

$$\coprod_{\alpha \in A} X_{\alpha}.$$

The coproduct topology on  $\coprod_{\alpha} X_{\alpha}$  has open sets

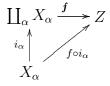
$$\{\coprod_{\alpha} U_{\alpha} : U_{\alpha} \subset X_{\alpha} \text{ is open in } X_{\alpha}\}$$

With this topology, each of the inclusions  $i_{\alpha} : X_{\alpha} \to \coprod X_{\alpha}$  is the inclusion of a subspace. The coproduct topology is characterized by

the following property: a function

$$f:\coprod_{\alpha} X_{\alpha\in A} \to Z$$

to a topological space Z is continuous if and only if the restriction  $f \circ i_{\alpha}$ of f to each  $X_{\alpha}$  is continuous:



If  $\{U_{\alpha} : \alpha \in A\}$  is an open covering of A, then the mapping

$$\coprod_{\alpha \in A} U_{\alpha} \to X$$

induced by the inclusions  $U_{\alpha} \hookrightarrow X$  is a quotient mapping.

If  $X = F_1 \cup F_2 \cup \cdots \cup F_n$ , where each  $F_j$  is a closed subset of X, then the mapping

 $F_1 \amalg F_2 \amalg \cdots \amalg F_n \to X$ 

is a quotient mapping. (It is important that n be finite here.) This statement is equivalent to the *glueing lemma*.

PRODUCT TOPOLOGY: Suppose that  $\{X_{\alpha} : \alpha \in A\}$  is a family of topological spaces indexed by the set A. The product topology on  $\prod_{\alpha} X_{\alpha}$  has basis the open sets:

 $\{\prod_{\alpha} U_{\alpha} : U_{\alpha} \subset X_{\alpha} \text{ is open in } X_{\alpha} \text{ and } X_{\alpha} = U_{\alpha} \text{ for almost all } \alpha\},\$ 

where "almost all" means "all but a finite number." Each of the projections  $p_{\alpha} : \prod_{\alpha} X_{\alpha} \to X_{\alpha}$  is a quotient mapping. The product topology is characterized by the following property: a function

$$f: Z \to \prod_{\alpha} X_{\alpha \in A}$$

from a topological space Z is continuous if and only if the  $\alpha$ th component  $p_{\alpha} \circ f : Z \to X_{\alpha}$  is continuous:

$$Z \xrightarrow{f} \prod_{\alpha} X_{\alpha}$$

$$\downarrow^{p_{\alpha}} \downarrow^{p_{\alpha}}$$

$$Z \xrightarrow{p_{\alpha} \circ f} \qquad \downarrow^{p_{\alpha}}$$

A common notation for f is  $f = (f_{\alpha})$ , where  $f_{\alpha} := p_{\alpha} \circ f$ . BASIC FACTS FROM TOPOLOGY:

- (i) A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.
- (ii) A compact subspace of a Hausdorff space is closed.
- (iii) The continuous image of a compact space is compact.