Richard Hain September 8, 2022

Math 611
Algebraic Topology I
Spheres as Quotients

The goal of this note is to sketch an efficient proof that the space obtained by identifying the boundary of the n-ball to a point is an n-sphere:

$$B^n/S^{n-1} \approx S^n.$$

We have already seen that B^n is the cone over its boundary S^{n-1} via the map

$$(S^{n-1} \times [0,1])/S^{n-1} \times \{0\} \to B^n$$

induced by $(x,t) \to tx$. This means that B^n/S^{n-1} is the quotient of $S^{n-1} \times [0,1]$ obtained by identifying $S^{n-1} \times \{0\}$ and $S^{n-1} \times \{1\}$ to points. By rescaling the interval, we see that B^n/S^{n-1} is the space obtained from $S^{n-1} \times [-1,1]$ by identifying $S^{n-1} \times \{-1\}$ and $S^{n-1} \times \{1\}$ to points. To complete the proof, we have to show that this is homeomorphic to S^n.

To this end, view S^n as the unit sphere $t^2 + \|x\|^2 = 1$ in $\mathbb{R} \times \mathbb{R}^n$ with coordinates (t,x). Define

$$S^{n-1} \times [-1,1] \to S^n$$

by $(x,t) \mapsto (t, \sqrt{1-t^2}\,x)$. This collapses $S^n \times \{\pm 1\}$ to the poles $(\pm 1, 0)$ and induces a continuous bijection

$$B^n/S^{n-1} \approx (S^{n-1} \times [-1,1])/\sim \quad \to S^n$$

where $(t,x) \sim (t',x')$ if and only if $t = t' = \pm 1$. It is therefore a homeomorphism.