November 4, 2022

Math 611
Quaternions Worksheet

1. The quaternions $\mathbb{H}$ is the four dimensional real vector space spanned by the linearly independent elements 1, i, j and k:

$$\mathbb{H} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{R}\}.$$

Define

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

Extend this to an $\mathbb{R}$-bilinear function

$$\mathbb{H} \times \mathbb{H} \to \mathbb{H}.$$

Show that, with this multiplication, $\mathbb{H}$ is an associative ring. Here and elsewhere, it may be helpful to use the complex representation

$$\mathbb{H} = \{z + wj : z, w \in \mathbb{C}\}.$$

Find a formula for the product of two quaternions written in this form.

2. Define the *conjugate* $\overline{q}$ of the quaternion q by

$$\overline{a + bi + cj + dk} = a - bi - cj - dk.$$

Show that $\overline{q_1 q_2} = \overline{q}_2 \overline{q}_1$. Define the *norm* $\|q\|$ of $q \in \mathbb{H}$ by

$$\|a + bi + cj + dk\| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

Check that $\|q\|^2 = q\overline{q}$. Deduce that $\|q_1 q_2\| = \|q_1\|\|q_2\|$ and that each non-zero quaternion is a unit. (I.e., has a multiplicative inverse.) Give a formula for the inverse. Denote the group of units in $\mathbb{H}$ by $\mathbb{H}^\times$.

3. Regard $\mathbb{H}$ as a $\mathbb{C}$ vector space by multiplication on the left. Show that for each $q \in \mathbb{H}$ the map

$$\Phi(q) : z + wj \mapsto (z + wj)\overline{q}$$

is $\mathbb{C}$-linear so that $\Phi(q) \in \mathrm{End}_{\mathbb{C}}(\mathbb{H})$. Show that the map

$$\Phi : \mathbb{H} \to \mathrm{End}_{\mathbb{C}}(\mathbb{H}), \quad q \mapsto \Phi(q)$$

is injective and preserves addition and multiplication. Deduce that $\mathbb{H}$ is an $\mathbb{R}$-algebra.

Identify $\mathbb{H}$ with $\mathbb{C}^2$ by identifying $(z, w) \in \mathbb{C}^2$ with $z + wj$. This identifies $\mathrm{End}_{\mathbb{C}}(\mathbb{H})$ with with $\mathbb{M}_2(\mathbb{C})$. With this convention, Φ is an algebra homomorphism $\mathbb{H} \to \mathbb{M}_2(\mathbb{C})$ which induces an injective group homomorphism $\Phi : \mathbb{H}^\times \to \mathrm{GL}_2(\mathbb{C})$.

4. Identify the set of unit quaternions (i.e., Quaternions of unit length) with S^3:

$$S^3 = \{q \in \mathbb{H} : \|q\| = 1\}.$$

Show that quaternion multiplication gives S^3 the structure of a group and that the homomorphism Φ above induces a group isomorphism $S^3 \to \mathrm{SU}(2)$. Our goal is to construct a homomorphism $S^3 \to \mathrm{SO}(3)$ whose kernel is $\pm I$.

5. The *real part* $\mathrm{Re}(q)$ of a quaternion q is defined by

$$\mathrm{Re}(a + bi + cj + dk) = a.$$

Purely imaginary quaternions $\mathrm{Im}\,\mathbb{H}$ are those with trivial real part. Think of $\mathrm{Im}\,\mathbb{H}$ as being the equatorial plane of S^3 and 1 as being the north pole, -1 as the south pole. Note that it makes sense to talk about the *latitude* of a point q on S^3 — namely the angle between 1 and q. To help you compute the latitude θ of an element q of S^3, note that that $\cos\theta = \mathrm{Re}(q)$. Also, show that if we define

$$(x, y) = \mathrm{Re}(x\overline{y}) = -\mathrm{Re}(xy)$$

for $x, y \in \mathrm{Im}\,\mathbb{H}$, then $(\ ,\ )$ is a positive definite inner product on $\mathrm{Im}\,\mathbb{H}$ and that i, j, k is an orthonormal basis of $\mathrm{Im}\,\mathbb{H}$.

6. Show that if $q \in S^3$ and $x \in \mathrm{Im}\,\mathbb{H}$, then $qxq^{-1} \in \mathrm{Im}\,\mathbb{H}$.[1] Define a homomorphism $A : S^3 \to \mathrm{GL}(\mathrm{Im}\,\mathbb{H})$ by

$$A(q) : x \mapsto qxq^{-1}.$$

Show that each $A(q)$ preserves the inner product, so that we have a homomorphism

$$A : S^3 \to \mathrm{O}(\mathrm{Im}\,\mathbb{H}) \cong \mathrm{O}(3).$$

Show that the kernel of A is $\pm I$.

7. We want to understand A and show that it has image $\mathrm{SO}(3)$, so that we have a short exact sequence

$$1 \to \{\pm I\} \to S^3 \to \mathrm{SO}(3) \to 1.$$

To this end, we define, for each $q \in \mathbb{H}$,

$$e^q = \sum_{n=0}^{\infty} \frac{q^n}{n!}.$$

Show that this series converges absolutely for each q. Note that $\exp(q_1 + q_2)$ does not always equal $\exp q_1 \exp q_2$; a sufficient condition for equality is that q_1 and q_2 commute. Show that

$$\|e^q\|^2 = e^q e^{\overline{q}} = e^{2\,\mathrm{Re}(q)}.$$

[1] The following formula may help. For $x, y \in \mathrm{Im}\,\mathbb{H}$, we have $xy = -(x, y) + x \times y$, where $\times$ denotes the cross product of vectors in $\mathbb{R}^3$, which we identify with $\mathrm{Im}\,\mathbb{H}$ in the natural way.

Deduce that if $q \in \operatorname{Im} \mathbb{H}$, then $e^q \in S^3$ so that the *exponential mapping*

$$\exp : \operatorname{Im} \mathbb{H} \to S^3 \quad q \mapsto e^q.$$

is well defined. Show that if $q \in \operatorname{Im} \mathbb{H}$ and $q \neq 0$, then

$$e^q = \cos \|q\| + \frac{q}{\|q\|} \sin \|q\|.$$

Deduce that e^q has latitude $\|q\|$ when $q \in \operatorname{Im} \mathbb{H}$ and that $\exp : \operatorname{Im} \mathbb{H} \to S^3$ is surjective.

8. Show that if $q \in \operatorname{Im} \mathbb{H}$, then $A(e^q)$ is the rotation about the line with axis q by an angle of twice the latitude of e^q — that is, by $2\|q\|$. Deduce that the image of S^3 under A is $\mathrm{SO}(3)$. (Hint: Consider the action of $e^y \in S^3$ on $x \in \mathbb{H}$ when x is a multiple of y, and then when x and y are perpendicular.)