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Math 611
Products of Quotient Maps

Suppose that $f : X \to Y$ and $g : Z \to W$ are quotient mappings. It is not always true that $f \times g : X \times Z \to Y \times W$ is a quotient map.[1] Here we give a general condition under which it is.

First recall that a continuous map $h : P \to Q$ is *proper* if the inverse image of each compact subset of Q is compact. We will prove that if Y is locally compact and g is proper, then $f \times g$ is a quotient mapping. Since the unit interval is compact, it will follow that "homotopies descend":

Proposition: *Suppose that $f : X \to Y$ is a quotient mapping and that $F : X \times I \to P$ is a homotopy. If $h : P \to Q$ is a continuous function and if F induces a function $G : Y \times I \to Q$ such that the diagram*

$$\begin{array}{ccc} X \times I & \xrightarrow{F} & P \\ {\scriptstyle f \times \mathrm{id}_I}\downarrow & & \downarrow{\scriptstyle h} \\ Y \times I & \xrightarrow{G} & Q \end{array}$$

commutes, then G is continuous. □

Corollary. *Every homotopy $F : (X \times I, A \times I) \to (Y, B)$ induces a homotopy $\overline{F} : (X/A) \times I \to Y/B$.* □

We will need the following basic fact:

Lemma. *If U is an open subset of the product $A \times B$ of topological spaces A and B, and if K is a compact subset of B, then*

$$V := \{a \in A : \{a\} \times K \subset U\}$$

is open in A.

Proof. If V is empty, there is nothing to prove. Suppose that $a_o \in V$. Since U is open, for each $k \in K$ we can find an open neighbourhood V_k of a_o in A and W_k of $k \in K$ such that $V_k \times W_k \subseteq U$. Since K is compact and since the W_k cover K, there is a finite subset F of K such that $K \subset \bigcup_{k \in F} W_k$. Then

$$V_o = \bigcap_{k \in F} V_k$$

is an open neighbourhood of a_o in A. Since $V_o \times K \subset U$. $V_o \subset V$. It follows that V is open in A. □

[1] The product of $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ and the identity $\mathbb{Q} \to \mathbb{Q}$ is not a quotient map.

The following result is due to J. H. C. Whitehead.

Proposition. (Whitehead [1, Lemma 4, p. 1131]) *Suppose that* $f : X \to Y$ *and* $g : Z \to W$ *are quotient mappings. If* W *is locally compact and* g *is proper, then* $f \times g : X \times Z \to Y \times W$ *is a quotient mapping.*

Proof. Since f and g are continuous, so is $f \times g : X \times Z \to Y \times W$. To prove the assertion, we need to show that if V is a non-empty subset of $Y \times W$ whose inverse image in $X \times Z$ is open, then V is open in $Y \times W$.

Suppose that $(y_o, w_o) \in V$. Pick (x_o, z_o) such that $f(x_o) = y_o$ and $g(z_o) = w_o$. Note that $\{x_o\} \times Z$ and $\{y_o\} \times W$ are homeomorphic to Z and W, respectively. Since the restriction of $f \times g$ to $\{x_o\} \times Z$ induces a quotient mapping $\{x_o\} \times Z \to \{y_o\} \times W$, and since

$$(\{x_o\} \times Z) \cap (f \times g)^{-1}(V)$$

is open in $\{x_o\} \times Z$, it follows that $(\{y_o\} \times W) \cap V$ is open in $\{y_o\} \times W$. Since W is locally compact, w_o has a compact neighbourhood K in W such that $\{y_o\} \times K \subset V$. Since g is proper, $g^{-1}(K)$ is compact. By the Lemma,

$$S := \{x \in X : \{x\} \times g^{-1}(K) \subseteq (f \times g)^{-1}(V)\}$$

is open in X. Since $x_o \in S$, it is non-empty. Since

$$x \in S \iff f(x) \times K \subseteq V$$

it follows that $S = f^{-1}(f(S))$ and that

$$f(S) = \{y \in Y : y \times K \subseteq V\}.$$

Since f is a quotient mapping, $f(S)$ is open in Y. Since $y_o \in f(S)$, $f(S) \times K$ is a neighbourhood of (y_o, w_o) in V. It follows that V is open in $Y \times W$. $\square$

References

[1] J. H. C. Whitehead: *Note on a theorem due to Borsuk*, Bull. Amer. Math. Soc. 54 (1948), 1125–1132.