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Math 611
Rough Notes on Group Actions

Suppose that G is a Hausdorff topological group, that K is a compact subgroup of G and that Γ is a discrete subgroup of G. We will show that the action of Γ on G/K by left multiplication is properly discontinuous.

1. Γ is closed in G: since Γ is a discrete subset of G, there is a neighbourhood U of the identity e in G such that $\Gamma \cap U = \{e\}$. Since the map $\phi : G \times G \to G$ defined by $\phi(g,h) = gh^{-1}$ is continuous, and takes (e,e) to e, there is a neighbourhood V of e in G such that $\phi(V \times V) \subset U$.

To show that Γ is closed in G, suppose that $g \in G - \Gamma$. Then gV is a neighbourhood of g in G. If $\gamma_1, \gamma_2 \in \Gamma \cap gV$, then there are $v_1, v_2 \in V$ such that $\gamma_j = gv_j$. But then $\gamma_1^{-1}\gamma_2 = v_1^{-1}v_2 \in V^{-1}V \subset U$, so that $\gamma_1^{-1}\gamma_2 \in \Gamma \cap U = \{1\}$ and $\gamma_1 = \gamma_2$. Thus $|\Gamma \cap gV| \leq 1$ and $gV - (gV \cap \Gamma)$ is an open neighbourhood of g in G.

This argument also shows that Γ acts properly discontinuously (and freely) on G. First observe that if $\gamma_1, \gamma_2 \in \Gamma$, then

$$\gamma_1 V \cap \gamma_2 V \neq \emptyset$$

implies that there are $v_1, v_2 \in V$ such that $\gamma_1 v_1 = \gamma_2 v_2$. That is, $\gamma_2^{-1}\gamma_1 = v_2 v_1^{-1} \in U$. This implies that $\gamma_1 = \gamma_2$.

Now suppose that $g \in G$. Then Vg is a neighbourhood of g. If $\gamma V g \cap V g$ is non-empty, then so is $\gamma V \cap V$, which implies that $\gamma = 1$.

2. K-saturated subsets of G: The subgroup K acts on G by right multiplication. For a subset X of G, set

$$XK := \bigcup_{x \in X} xK = \{xk : x \in X,\ k \in K\}.$$

A subset X of G is K-saturated if $X = XK$. Equivalently, it is K-saturated if it is a union of K-orbits. Let $p : G \to G/K$ be the projection. Note that $A \mapsto p^{-1}(A)$ defines a 1-1 correspondence between (open, closed, arbitrary, ...) subsets of G/K and (open, closed, arbitrary, ...) K-saturated subsets of G.

Since K is compact, every neighbourhood W of K in G contains a K-saturated neighbourhood. To see this, suppose that $k \in K$. Since the multiplication map $G \times K \to G$ is continuous, and since $(1,k) \mapsto k$, there is an open neighbourhood $U_k \times N_k$ of $(1,k)$ in $G \times K$ such that

$U_k N_k \subset W$. Since K is compact, the open covering $\{N_k : k \in K\}$ of K has a finite subcovering $N_{k_1}, \dots, N_{k_n}$. Then $U := \bigcap_{j=1}^{n} U_{k_j}$ is an open neighbourhood of e in G and UK is a K-saturated open neighbourhood of K in G that is contained in W.

Note that the saturated open neighbourhood UK of K in G corresponds to an open neighbourhood of the identity coset in G/K.

3. Γ acts properly discontinuously on G/K: Since Γ is closed in G (by the first part), $G - (\Gamma - \Gamma \cap K)$ is an open neighbourhood of K in G. Fix a K-saturated neighbourhood U of K that is contained in $G - (\Gamma - \Gamma \cap K)$. It exists by the second part and has the property that $\Gamma \cap U = \Gamma \cap K$. We first show that K has a saturated open neighbourhood V such that if $v_1, v_2 \in V$, then $v_1 v_2^{-1} \in U$. For this we use the map $\phi : G \times G \to G$ that takes (g, h) to gh^{-1}. Since ϕ is continuous, $\phi^{-1}(U)$ is an open neighbourhood of $K \times K$ in $G \times G$. By a compactness argument similar to the one in the previous paragraph, there is an open neighbourhood W of $e \in G$ and an open neighbourhood N of K in G such that

$$\phi(N \times W) = NW^{-1} \subseteq U.$$

By the second part, we may assume that N is K-saturated, so that

$$\phi(N \times Wk) = N(Wk)^{-1} = Nk^{-1}W^{-1} = NW^{-1} \subseteq U$$

for all $k \in K$. In other words, $N \times Wk \subseteq \phi^{-1}(U)$ for all $k \in K$.

Set $V = N \cap WK$. This is the intersection of two K-saturated sets and is therefore a K-saturated neighbourhood of K in G. The calculation above implies that

$$\phi(V \times V) \subseteq \phi(N \times W) \subset U.$$

as required.

Now, if $\gamma \in \Gamma$, then $\gamma V \cap V$ is non-empty implies that there are $v_1, v_2 \in V$ such that $\gamma v_1 = v_2$, so that $\gamma = v_2 v_1^{-1} \in U$. This implies that $\gamma \in \Gamma \cap U = \Gamma \cap K$. Thus

$$\gamma V \cap V \neq \emptyset \Leftrightarrow \gamma \bar{e} = \bar{e} \text{ in } G/K.$$

The general case follows by replacing K by gKg^{-1}. Take V to be a gKg^{-1}-saturated neighbourhood of gKg^{-1} in G such that $VV^{-1} \cap \Gamma = \Gamma \cap gKg^{-1}$, which is the stabilizer of gK in Γ. Then Vg is a K-saturated neighbourhood of gK in G and

$$\gamma Vg \cap Vg \neq \emptyset \Leftrightarrow \gamma V \cap V \neq \emptyset \Leftrightarrow \gamma \in \Gamma \cap gKg^{-1}.$$