Richard Hain December 7, 2022

Math 611
Problem Set 6

Due: 5 pm, Saturday December 17, 2022

1. Suppose that if X is a topological space that is the union $X = A \cup B$ of two open subsets. Suppose that R is a PID and that A, B, $A \cup B$ and $A \cap B$ are homotopy equivalent to finite complexes. Show that

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

Hint: Use Mayer-Vietoris and the fact that the Euler characteristic of an exact sequence is zero.

2. Suppose that S is a compact oriented surface of genus g and that $P = \{x_1, \dots, x_n\}$ is a finite subset of S. Set $S' = S - P$. This is commonly referred to as a surface of type (g, n). Compute the Euler characteristic $\chi(S')$ of S'. For which (g, n) is $\chi(S') < 0$?

Kultcha: the importance of this computation is that S' has a complete metric of constant curvature -1 if and only if $\chi(S') < 0$. Such surfaces are said to be *hyperbolic.*

3. A *pair of pants* is a compact orientable surface P of genus 0 with 3 boundary components. Alternatively, it is a "disk with two holes".

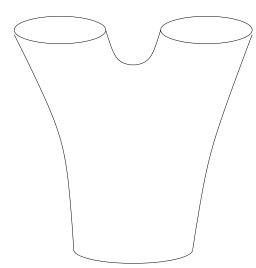

Figure 1. A pair of pants

Compute $\chi(P)$.

The interior $P^o := P - \partial P$ is referred to an *open pair of pants.*

4. A *pants decomposition* of a compact orientable surface X of genus g is a collection of disjoint simple closed curves $C_1, \dots, C_N$ in X such that

$$X - \bigcup_{j=1}^{N} C_j$$

is a disjoint union of open pairs of pants. Show that a closed, connected surface has a pants decomposition if and only if its genus is > 1 and that, in this case, $N = 3g - 3$.

5. Suppose that X is a topological space (Hausdorff, as always) and that $\phi : X \to X$ is a homeomorphism. The *mapping torus* W of ϕ is defined to be

$$(X \times [0,1])/\{(x,1) \sim (\phi(x), 0) : x \in X\}.$$

Note that the projection $X \times [0,1] \to [0,1]$ induces a projection $p : W \to S^1$.

(i) Show that the action

$$n : (x,t) \mapsto (\phi^{-n}(x), t+n) \qquad n \in \mathbb{Z}$$

of $\mathbb{Z}$ on $X \times \mathbb{R}$ is properly discontinuous and fixed point free.[1]

(ii) Show that the W is homeomorphic to the quotient of $X \times \mathbb{R}$ by this $\mathbb{Z}$-action and that the projection $X \times \mathbb{R} \to \mathbb{R}$ induces the projection p. (For simplicity, you may assume that X is compact.)

(iii) Show that if X is path connected, there is a natural short exact sequence

$$1 \to \pi_1(X, x) \to \pi_1(W, x) \xrightarrow{p_*} \pi_1(S^1, p(x)) \to 1$$

where (X, x) is regarded as the fiber $p^{-1}(1)$ of p over $1 \in S^1$. (Hint: use a result about fundamental groups of quotients by properly discontinuous and free group actions.)

6. View the torus T as $\mathbb{R}^2/\mathbb{Z}^2$. By covering space theory, we can identify $\pi_1(T, 0)$ with $\mathbb{Z}^2$. Since $\pi_1(T, 0) = H_1(T; \mathbb{Z})$, we can identify $\operatorname{Aut} H_1(T; \mathbb{Z})$ with $\mathrm{GL}_2(\mathbb{Z})$.

(i) Show that the linear action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{R}^2$ induces a homomorphism

$$\rho : \mathrm{GL}_2(\mathbb{Z}) \to \operatorname{Aut}(T)$$

such that $\rho(\phi)_* = \phi$ in $\operatorname{Aut} H_1(T; \mathbb{Z})$.

(ii) Fix a prime number p. Compute the homology $H_\bullet(W; R)$ of the mapping torus W of

$$\phi = \begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}.$$

[1] Recall that a group action is *fixed point free* (or simply *free*) if $gx = x$ implies that $g = 1$, where $g \in G$ and $x \in X$.

Here you should compute the answer for $R = \mathbb{Z}$ and $R = \mathbb{Z}/q\mathbb{Z}$ for each prime number q. (Hint: Use Mayer-Vietoris.) You may assume that ϕ induces the identity on $H_2(T)$, but you will receive more credit is you prove this.