Richard Hain October 31, 2022 (updated November 6)

MATH 611
PROBLEM SET 4

Due: Tuesday, November 14, 2022

1. Suppose that X is a connected topological space. A covering map $p : Y \to X$ is an abelian covering of X if it is a (connected) Galois covering of X and $\mathrm{Aut}(Y/X)$ is abelian. Show that if X is nice (e.g., locally simply connected), then X has an abelian covering $a : X' \to X$ which dominates all other abelian coverings: if $Y \to X$ is an abelian covering, then there is a covering mapping $q : X' \to Y$ such that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{q} & Y \\ & {\scriptstyle a}\searrow & \downarrow{\scriptstyle p} \\ & & X \end{array}$$

commutes. The covering $a : X' \to X$ is called a *maximal abelian covering* of X. Prove that any two maximal abelian coverings of X are isomorphic. (Often we will abuse terminology and refer to *the* maximal abelian covering of X.) Show that A-invariant isomorphism classes of abelian coverings of X with Galois group A correspond to surjective homomorphisms $\phi : H_1(X;\mathbb{Z}) \to A$.[1]

2. Consider the bouquet

$$X = \bigvee_{j=1}^{n} S^1$$

of n circles. Let $j : X \to (S^1)^n$ be the standard inclusion. Show that the homomorphism

$$j_* : \pi_1(X, x_o) \to \pi_1\big((S^1)^n, 0\big)$$

is surjective and that $\pi_1\big((S^1)^n, 0\big)$ is the maximal abelian quotient of $\pi_1(X, x_o)$. Deduce that the inverse image Y of X in $\mathbb{R}^n$ under the covering map $\mathbb{R}^n \to (S^1)^n$ is connected and that the projection a :

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[1] Suppose that $Y \to X$ and $Z \to X$ are abelian coverings with Galois group A. An isomorphism $f : Y \to Z$ is A-invariant if $f(ay) = af(y)$.

$Y \to X$ is the maximal abelian covering of X.

$$\begin{array}{ccc} Y & \xhookrightarrow{J} & \mathbb{R}^n \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[j]{} & (S^1)^n \end{array}$$

(i) Show that there is a homeomorphism

$$Y \cong \bigcup_{k=1}^{n} \mathbb{Z} \times \cdots \times \mathbb{Z} \times \overset{k}{\overset{|}{\mathbb{R}}} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \subset \mathbb{R}^n.$$

Deduce that $\pi_1(Y, y)$ is a countably generated free group when $n > 1$.

(ii) Show that $H_1(Y; \mathbb{Z})$ is free abelian group generated by the same set.

(iii) Show that $H_1(Y; \mathbb{Z})$ is a module over the group algebra of $H_1(X; \mathbb{Z})$. Note that this group algebra is isomorphic to the ring

$$\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

of Laurent polynomials in $t_1, \ldots, t_n$.

(iv) Show that $H_1(Y; \mathbb{Z})$ is finitely generated as a $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ module. (Suggestion: first try proving this when $n = 2$ and/or 3.)

3. Show that SO(3) is homeomorphic to $\mathbb{RP}^3$. There are several ways to approach this; the easiest is to use quaternions. Note that the set of unit quaternions is a group which is homeomorphic to S^3. Identify $\mathbb{R}^3$ with the space of imaginary quaternions

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \quad (x, y, z) \in \mathbb{R}^3.$$

Define a homomorphism $\varphi : S^3 \to \mathrm{GL}_3(\mathbb{R})$ from the group of unit quaternions to $\mathrm{GL}_3(\mathbb{R})$ by $\varphi(q) : v \mapsto qvq^{-1}$.

(i) Show that for each $q \in S^3$, $\varphi(q)$ preserves lengths. Deduce that the image of φ is contained in O(3). Explain why $\operatorname{im} \varphi \subseteq \mathrm{SO}(3)$. (Hint: use the quaternion norm and the fact that $\|q_1 q_2\| = \|q_1\| \|q_2\|$.)

(ii) Show that $\ker \varphi = \{\pm 1\}$.

Deduce that $\pi_1(\mathrm{SO}(3), 1) \cong \mathbb{Z}/2\mathbb{Z}$. You may use the fact that every neighbourhood of the identity of a connected Lie group (such as SO(3)) generates the group.

Remark: One can use this and the fact that $\mathrm{SO}(n+1)/\mathrm{SO}(n) \approx S^n$ is simply connected when $n \geq 2$ to prove that $\pi_1(\mathrm{SO}(n+1), 1) \cong \pi_1(\mathrm{SO}(n), 1) \cong \mathbb{Z}/2\mathbb{Z}$ for all $n \geq 3$.