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Math 611 Problem Set 4

Due: Tuesday, November 14, 2022

1. Suppose that X is a connected topological space. A covering map $p: Y \to X$ is an abelian covering of X if it is a (connected) Galois covering of X and $\operatorname{Aut}(Y|X)$ is abelian. Show that if X is nice (e.g., locally simply connected), then X has an abelian covering $a: X' \to X$ which dominates all other abelian coverings: if $Y \to X$ is an abelian covering, then there is a covering mapping $q: X' \to Y$ such that the diagram



commutes. The covering $a : X' \to X$ is called a *maximal abelian* covering of X. Prove that any two maximal abelian coverings of X are isomorphic. (Often we will abuse terminology and refer to the maximal abelian covering of X.) Show that A-invariant isomorphism classes of abelian coverings of X with Galois group A correspond to surjective homomorphisms $\phi : H_1(X; \mathbb{Z}) \to A^{1}$

2. Consider the bouquet

$$X = \bigvee_{j=1}^{n} S^{1}$$

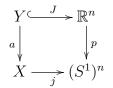
of n circles. Let $j: X \to (S^1)^n$ be the standard inclusion. Show that the homomorphism

$$j_*: \pi_1(X, x_o) \to \pi_1((S^1)^n, 0)$$

is surjective and that $\pi_1((S^1)^n, 0)$ is the maximal abelian quotient of $\pi_1(X, x_o)$. Deduce that the inverse image Y of X in \mathbb{R}^n under the covering map $\mathbb{R}^n \to (S^1)^n$ is connected and that the projection a:

¹Suppose that $Y \to X$ and $Z \to X$ are abelian coverings with Galois group A. An isomorphism $f: Y \to Z$ is A-invariant if f(ay) = af(y).

 $Y \to X$ is the maximal abelian covering of X.



(i) Show that there is a homeomorphism

$$Y \cong \bigcup_{k=1}^{n} \mathbb{Z} \times \cdots \times \mathbb{Z} \times \mathbb{R}^{k} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \subset \mathbb{R}^{n}.$$

Deduce that $\pi_1(Y, y)$ is a countably generated free group when n > 1.

- (ii) Show that $H_1(Y;\mathbb{Z})$ is free abelian group generated by the same set.
- (iii) Show that $H_1(Y;\mathbb{Z})$ is a module over the group algebra of $H_1(X;\mathbb{Z})$. Note that this group algebra is isomorphic to the ring

$$\mathbb{Z}[t_1^{\pm 1},\ldots,t_n^{\pm 1}]$$

of Laurent polynomials in t_1, \ldots, t_n .

(iv) Show that $H_1(Y;\mathbb{Z})$ is finitely generated as a $\mathbb{Z}[t_1^{\pm 1},\ldots,t_n^{\pm 1}]$ module. (Suggestion: first try proving this when n = 2 and/or 3.)

3. Show that SO(3) is homeomorphic to \mathbb{RP}^3 . There are several ways to approach this; the easiest is to use quaternions. Note that the set of unit quaternions is a group which is homeomorphic to S^3 . Identify \mathbb{R}^3 with the space of imaginary quaternions

$$x\mathbf{i} + y\mathbf{j} + z\mathbf{k}, \quad (x, y, z) \in \mathbb{R}^3.$$

Define a homomorphism $\varphi : S^3 \to \operatorname{GL}_3(\mathbb{R})$ from the group of unit quaternions to $\operatorname{GL}_3(\mathbb{R})$ by $\varphi(q) : v \mapsto qvq^{-1}$.

- (i) Show that for each $q \in S^3$, $\varphi(q)$ preserves lengths. Deduce that the image of φ is contained in O(3). Explain why im $\varphi \subseteq$ SO(3). (Hint: use the quaternion norm and the fact that $||q_1q_2|| = ||q_1|| ||q_2||$.)
- (ii) Show that ker $\varphi = \{\pm 1\}$.

Deduce that $\pi_1(SO(3), 1) \cong \mathbb{Z}/2\mathbb{Z}$. You may use the fact that every neighbourhood of the identity of a connected Lie group (such as SO(3)) generates the group.

Remark: One can use this and the fact that $SO(n + 1)/SO(n) \approx S^n$ is simply connected when $n \geq 2$ to prove that $\pi_1(SO(n + 1), 1) \cong \pi_1(SO(n), 1) \cong \mathbb{Z}/2\mathbb{Z}$ for all $n \geq 3$.