Richard Hain September 30, 2022

Math 611
Problem Set 3

Due: Thursday, October 20, 2022

1. Suppose that $p : Y \to X$ is a covering map and that Δ is a contracible subset of X. Show that any two distinct lifts of Δ to Y are disjoint.

2. A *solid torus* is a topological space that is homeomorphic to $S^1 \times B^2$. Consider S^3 to be the unit sphere in $\mathbb{C}^2$:

$$S^3 = \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 = 1\}.$$

Fix a real number a satisfying $0 < a < 1$. Let

$$T(a) = \{(x, y) \in S^3 : |x|^2 = a\},$$

$$U_1(a) = \{(x, y) \in S^3 : |x|^2 \leq a\} \text{ and } U_2(a) = \{(x, y) \in S^3 : |x|^2 \geq a\}.$$

Show that $T(a)$ is a 2-torus and that $U_1(a)$ and $U_2(a)$ are solid tori that intersect in $T(a)$. Deduce that S^3 is homeomorphic to

$$U_1(a) \cup_{T(a)} U_2(a) := (U_1(a) \amalg U_2(a))/\sim$$

where the equivalence relation $\sim$ identifies $x \in U_1(a)$ with $y \in U_2(a)$ if and only if $x = y \in T(a)$.

3. As above, view S^3 as the unit sphere in $\mathbb{C}^2$. Let

$$L_1 = \{(x, y) \in S^3 : x = 0\} \text{ and } L_2 = \{(x, y) \in S^3 : y = 0\}.$$

Show that L_1 and L_2 are disjoint imbedded circles in S^3. Show that $T(a)$ is a deformation retract of $S^3 - (L_1 \cup L_2)$. Use this to show that $\pi_1(S^3 - (L_1 \cup L_2), x_o)$ is isomorphic to $\mathbb{Z}^2$.

4. Suppose that m and n are positive integers. Denote the affine curve $x^m = y^n$ in $\mathbb{C}^2$ by C. Set $L = C \cap S^3$. For $\lambda \in \mathbb{C}$, define

$$\lambda \cdot (x, y) = (\lambda^n x, \lambda^m y)$$

(i) Show that the function $f : \mathbb{R}_{\geq 0} \times S^3 \to \mathbb{C}^2$ defined by $f(t, \xi) = t \cdot \xi$ induces a homeomorphism

$$(\operatorname{cone}(S^3), \operatorname{cone} L, 0) \to (\mathbb{C}^2, C, 0)$$

and a homeomorphism

$$\mathbb{R}_{>0} \times (S^3 - L) \to \mathbb{C}^2 - C.$$

(ii) Show that L is imbedded in S^3 as

$$\{(\theta,\phi) \in S^1 \times S^1 : m\theta \equiv n\phi \mod 2\pi\}$$

where $S^1 \times S^1$ denotes the torus

$$\{(x,y) \in S^3 : |x|^2 = a \text{ and } |y|^2 = 1 - a\}$$

for some suitable a satisfying $0 < a < 1$. (We say that L is a *torus link* of type (m,n).)

(iii) Show that the number of connected components of L is the greatest common divisor of m and n.

(iv) Suppose that $x \in S^3 - L$. Show that the inclusion $j : S^3 - L \hookrightarrow \mathbb{C}^2 - C$ induces an isomorphism

$$j_* : \pi_1(S^3 - L, x) \to \pi_1(\mathbb{C}^2 - C, x).$$