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Problem Set 2

Due: Tuesday October 4, 2022

1. Construct a natural group isomorphism

$$\pi_1\big(X \times Y, (x, y)\big) \xrightarrow{\simeq} \pi_1(X, x) \times \pi_1(Y, y).$$

2. Denote by C the set of free homotopy classes of maps $S^1 \to X$. Identify the quotient space $[0, 1]/(0 \sim 1)$ with S^1. Define a function

$$\Phi : \pi_1(X, x) \to C$$

by taking the homotopy class of a loop $\alpha : ([0, 1], \{0, 1\}) \to (X, x)$ to the free homotopy class of the induced mapping $S^1 \to X$. Show that if X is path connected, then Φ is surjective and that $\Phi(a) = \Phi(b)$ if and only if a and b are conjugate elements of $\pi_1(X, x)$. Deduce that C is the set of conjugacy classes of $\pi_1(X, x)$.

3. Show that if G is a topological group, then $\pi_1(G, e)$ is abelian. Hint: given loops α and β in G based at the identity e, define

$$F : I \times I \to G$$

by $F(s, t) = \alpha(s)\beta(t)$, where the product is taken in G. Use this to construct a homotopy from $\alpha \cdot \beta$ to $\beta \cdot \alpha$. For later use, it is also worth defining $\alpha * \beta$ to be $t \mapsto F(t, t)$ and showing that $\alpha\beta \simeq \alpha * \beta \simeq \beta\alpha$.

4. Suppose that $f, g : X \to Y$ are homotopic maps. Show that for each $x \in X$, there is a path γ in Y from $f(x)$ to $g(y)$ such that the diagram

$$\begin{array}{ccc} \pi_1(X, x) & \xrightarrow{f_*} & \pi_1(Y, f(x)) \\ & \searrow^{g_*} & \uparrow \Phi_\gamma \\ & & \pi_1(Y, g(x)) \end{array}$$

commutes. Use this to show that if X and Y are homotopy equivalent path connected spaces, then $\pi_1(X, x)$ is isomorphic to $\pi_1(Y, y)$ for all $x \in X$ and $y \in Y$.

5. Suppose that G is a Hausdorff topological group and that K is a compact subgroup. Set $X = G/K$.

(i) Show that if Γ is a discrete subgroup of G, then Γ is closed in G.[1]

[1] That is, Γ is a subset of G and, with the induced topology, Γ is a discrete topological space. Note that discrete subsets of a topological space may not be

(ii) Show that the stabilizer (i.e., isotropy group) Γ_x of $x = gK \in X$ is $\Gamma_x = \Gamma \cap gKg^{-1}$. Deduce that Γ_x is finite. (Note that it is trivial when K is trivial.)

(iii) It is a fact that if each Γ_x is trivial (e.g., Γ is torsion free[2]), then $p : X \to \Gamma\backslash X$ is a covering projection. Prove this in the case when K is trivial.

6. Show that $\mathrm{U}(n)$ is a deformation retract of $\mathrm{GL}_n(\mathbb{C})$ and that $\mathrm{SU}(n)$ is a deformation retract of $\mathrm{SL}_n(\mathbb{C})$.[3] Hint: use the matrix factorization that you get from Gram-Schmidt.

7. Show that $\mathrm{SL}_n(\mathbb{C})/\mathrm{SU}(n)$ is homeomorphic to the space X of unimodular, positive definite, hermitian matrices.[4] Deduce that if $d > 0$, then $\mathrm{SL}_n(\mathbb{Z}[\sqrt{-d}])$ acts properly discontinuously on X. Here you may assume that a discrete subgroup Γ and K a compact subgroup of a topological group G acts properly discontinuously on G/K.

8. Suppose that $p : (G', e') \to (G, e)$ is a connected covering of a connected, locally path connected topological group G. Show that G' has the structure of a topological group with identity e' for which the covering projection $(G', e') \to (G, e)$ is a group homomorphism. (Hint: Use lifting properties of, for example, the multiplication $(G \times G, (e, e)) \to (G, e)$.)

9. Show that if G is a path connected topological group and that $p : G' \to G$ is a connected covering, then the kernel of p is a discrete central subgroup of G'. Show that if $\widetilde{G} \to G$ is a universal covering of G, then $\ker p$ is naturally isomorphic to $\pi_1(G, e)$. (This gives a second solution to problem 3.)

10. Show that $\pi_1(\mathrm{SL}_2(\mathbb{R}), e) \cong \mathbb{Z}$ and deduce that the universal covering group $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ is an extension

$$0 \to \mathbb{Z} \to \widetilde{\mathrm{SL}}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R}) \to 1.$$

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closed. For example, $\{1/n : n \in \mathbb{Z},\ n \geq 1\}$ is a discrete subset of $\mathbb{R}$, but it is not closed.

[2]A theorem of Minkowski implies that the finite index subgroup

$$\{A \in \mathrm{SL}_n(\mathbb{Z}) : A \equiv I \bmod m\}$$

of $\mathrm{SL}_n(\mathbb{Z})$ is torsion free for all $m \geq 3$.

[3]A slightly simpler argument shows that $\mathrm{O}(n)$ is a deformation retract of $\mathrm{GL}_n(\mathbb{R})$ and that $\mathrm{SO}(n)$ is a deformation retract of $\mathrm{SL}_n(\mathbb{R})$. Details left to the reader!

[4]A similar argument can be used to show that $SL_n(\mathbb{R})/SO(n)$ is homeomorphic to the space of positive definite real symmetric matrices.