Richard Hain September 3, 2022

Math 611
Algebraic Topology I
Problem Set 1

Due: Tuesday, September 13, 2022

1. Show that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism.

2. Suppose that the group Γ acts continuously on the topological space X. Show that the quotient mapping $X \to \Gamma\backslash X$ from X to the orbit space is an open mapping when $\Gamma\backslash X$ is given the quotient topology. Give an example of a quotient mapping that is not open. (Hint: view the circle as a quotient of the interval.)

3. Show that the composite

$$S^n \hookrightarrow \mathbb{R}^{n+1} - \{0\} \to \mathbb{RP}^n$$

of the inclusion with the canonical quotient mapping induces a homeomorphism $S^n/\sim_S \to \mathbb{RP}^n$, where $x \sim_S y$ if and only if $x = \pm y$. Deduce that $\mathbb{RP}^n$ is compact. Define $f : B^n \to S^n$ by

$$f(x) = (\sqrt{1 - \|x\|^2}, x) \in \mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}.$$

Define an equivalence relation $\sim_B$ on B^n by $x \sim_B y$ if and only if $\|x\| = \|y\| = 1$ and $x = \pm y$. Show that the inclusion $B^n \to S^n$ induces homeomorphisms

$$B^n/\sim_B \to S^n/\sim_S \to \mathbb{RP}^n.$$

Hint: the easiest way to construct the inverse is to first construct a map $\mathbb{R}^{n+1} - \{0\} \to S^n$.

4. The circle S^1 acts continuously on the unit sphere

$$S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} : |z_0|^2 + \cdots + |z_n|^2 = 1\}$$

in $\mathbb{C}^{n+1}$ by left multiplication:

$$\lambda \cdot (z_0, \dots, z_n) = (\lambda z_0, \dots, \lambda z_n), \qquad \lambda \in S^1.$$

Prove that $\mathbb{CP}^n$ is homeomorphic to the quotient space S^{2n+1}/S^1.

5. Suppose that $\{(X_\alpha, x_\alpha) : \alpha \in A\}$ is a set of pointed spaces indexed by the set A (typically finite). The *wedge* of the (X_α, x_α) is defined to be

$$\bigvee_{\alpha \in A} X_\alpha := (\textstyle\coprod_{\alpha \in A} X_\alpha)/\sim$$

endowed with the quotient topology, where the equivalence relation identifies all x_α to a single point. The wedge of n copies of $(S^1, 1)$ is called a *bouquet of n circles.*

There is a natural inclusion

$$j : \bigvee_{\alpha \in A} X_\alpha \hookrightarrow \prod_{\alpha \in A} X_\alpha$$

For $y \in X_\alpha$ define $j(y) = (y_\beta)$ to be

$$y_\beta = \begin{cases} y & \text{if } \beta = \alpha; \\ x_\beta & \text{if } \beta \neq \alpha. \end{cases}$$

Show that j is the inclusion of a subspace when A is finite. (That is, the wedge of the X_α has the subspace topology induced from the product via j.)

6. Show that $(S^1)^2/(S^1 \vee S^1)$ is homeomorphic to S^2.

7. Suppose that K is a compact Hausdorff space. Show that if $f : X \to Y$ is a quotient mapping, then $f \times \mathrm{id}_K : X \times K \to Y \times K$ is also a quotient mapping.

8. Suppose that $x \in S^1 \times S^1$ is a point that does not lie in the image $S^1 \vee S^1 \hookrightarrow S^1 \times S^1$. Show that $S^1 \vee S^1$ is a deformation retraction of $(S^1 \times S^1) - \{x\}$. Hint: a good way to do this is to consider $S^1 \times S^1$ to be a quotient of the square I^2, which can be viewed as a cone over its boundary ∂I^2 with vertex x.