Richard Hain December 6, 2006

Math 272
Riemann Surfaces
Problem Set 4

Due: Friday, December 15, 2006.

1. Suppose that X is a compact Riemann surface and that F is a non-empty finite subset of X. Show that there is an embedding $\phi : X \hookrightarrow \mathbb{P}^N$ and a hyperplane H in $\mathbb{P}^N$ such that $H \cap X = F$. By choosing coordinates $[x_0, \dots, x_N]$ on $\mathbb{P}^N$ such that H has equation $x_0 = 0$, show that $X - F$ is a closed subvariety of $\mathbb{C}^N$. (With a little work, one can show that the image of X in $\mathbb{P}^N$ is cut out by (homogeneous) polynomial equations. This exercise thus shows that $X - F$ is cut out by polynomial equations in $\mathbb{C}^N$. That is, $X - F$ is an affine algebraic curve.)

2. Show that if X is a compact Riemann surface and if $P \in X$, then there is a meromorphic function $t \in \mathcal{M}(X)$ that vanishes to order 1 at P. (This is an algebraic local parameter at $P \in X$, which is defined on the Zariski neighbourhood $X - t^{-1}(\infty)$ of P in X.) Hint: use Riemann-Roch to prove the existence of $1/t$.

3. Suppose that X is a compact Riemann surface. Show that the pairing

$$w \otimes \xi \mapsto i \int_X w \wedge \bar{\xi}$$

defines a positive definite Hermitian form on $H^0(X, \Omega^1_X)$. Show that if $\phi \in \operatorname{Aut} X$, then $\phi^* : H^0(X, \Omega^1_X) \to H^0(X, \Omega^1_X)$ is an isometry.

4. Show that if X is a compact Riemann surface of genus $g \geq 2$, then the natural homomorphism

$$\operatorname{Aut}(X) \to \operatorname{Aut} H^0(X, \Omega^1_X)$$

is injective. Hints:

(i) Show that if X is not hyperelliptic and P, Q are distinct points of X, then there is a holomorphic differential on X that vanishes at P, but not at Q.

(ii) Show that if X is hyperelliptic, then the "hyperelliptic involution" acts as -1 on $H^0(X, \Omega^1_X)$. Show that if P and Q are distinct points of X, then either $\sigma(P) = Q$ or there is a holomorphic differential that vanishes at P and not at Q.

5. Show that if X is a compact Riemann surface of genus $g \geq 2$, then $\operatorname{Aut} X$ is finite. Hints:

(i) Show that there is a homomorphism

$$U(H^0(X, \Omega^1_X)) \hookrightarrow \operatorname{Aut} H^1(X; \mathbb{C})$$

defined by

$$A \mapsto (A, \bar{A}) \in \operatorname{Aut} \Omega^1(X) \times \operatorname{Aut} \overline{\Omega}^1(X) \subset \operatorname{Aut} \big(\Omega^1(X) \oplus \overline{\Omega}^1(X)\big).$$

Observe that this has compact image as unitary groups are compact.

(ii) Show that if $\phi : X \to X$ is a homeomorphism, then

$$\phi^* \in \operatorname{Aut} H^1(X; \mathbb{Z})$$

which is a discrete subgroup of $\operatorname{Aut} H^1(X; \mathbb{C})$.

(iii) Show that $U(H^0(X, \Omega^1_X)) \cap \operatorname{Aut} H^1(X; \mathbb{Z})$ is finite, where the intersection is taken inside $\operatorname{Aut} H^1(X; \mathbb{C})$.