

MATH 272
RIEMANN SURFACES
PROBLEM SET 3

Due: Tuesday, November 27, 2006.

1. Suppose that \mathcal{F} is a sheaf of abelian groups on a topological space X . Show that if U is an open subset of X , then $f \in \mathcal{F}(U)$ is zero if and only if the germ $f_x \in \mathcal{F}_x$ of f at each $x \in X$ vanishes.
2. Suppose that X is a Riemann surface. Define the presheaf $\exp \mathcal{O}_X$ on X by

$$\exp \mathcal{O}_X(U) = \{e^g : g \in \mathcal{O}_X(U)\}.$$

Show that this presheaf is not a sheaf as it does not satisfy the patching condition (II). Show that it does, however, satisfy sheaf axiom (I).

Show that the presheaf

$$U \mapsto \mathcal{O}_X^\times(U) / \exp \mathcal{O}_X(U)$$

is not a sheaf as it does not satisfy the vanishing axiom (I).

3. Suppose that F is a field and that

$$0 \longrightarrow C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} \cdots \xrightarrow{d} C^m \xrightarrow{d} 0$$

is a finite dimensional complex. (That is $d^2 = 0$ and each C^k is finite dimensional.) Show that

$$\sum_{k=0}^m (-1)^k \dim C^k = \sum_{k=0}^m (-1)^k \dim H^k(C^\bullet).$$

In particular, if the complex above is a long exact sequence, then

$$\sum_{k=0}^m (-1)^k \dim C^k = 0.$$

4. Suppose that U is a connected open subset of \mathbb{C} and A is a discrete subset of U . Suppose that for each $a \in A$, we have a “principal part”

$$P_a(z) = \sum_1^{N_a} c_k (z - a)^{-k}.$$

Show that there is $f \in \mathcal{M}(U)$ that is holomorphic on $U - A$ and has principal part $P_a(z)$ at each $a \in A$.

5. Suppose that X is a Riemann surface. Show that

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\pi i} \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^\times \longrightarrow 1$$

is an exact sequence of sheaves on X . Show that if U is a domain (i.e., a connected open subset of \mathbb{C}), then $H^1(U, \mathcal{O}_U^\times) = 0$. Use this to prove that if D is a divisor on U , then there is a meromorphic function f on U such that $(f) = D$. That is, every divisor on U is principal.

6. Show that if X is a compact Riemann surface, then $H^k(X, \Omega^p(D))$ vanishes for all divisors D , all $k > 1$ and $p = 0, 1$.