Richard Hain October 3, 2006 (revised, October 7, 8)

MATH 272
RIEMANN SURFACES
PROBLEM SET 2 (REVISED)

Due: Tuesday, October 17, 2006.

1. Suppose that X is a Riemann surface and that A is a closed discrete subset of X. Set $X' = X - A$. Suppose that Y' is another Riemann surface. Show that if $f : Y' \to X'$ is an unbranched proper covering map, then there is a Riemann surface Y, a closed discrete subset B of Y, and a proper holomorphic mapping $F : Y \to X$, where $Y' = Y - B$ and the restriction of F to Y' is f. Show that the restriction mapping

$$\operatorname{Aut}(Y/X) \to \operatorname{Aut}(Y'/X')$$

is an isomorphism.

2. Suppose that $f : X \to Y$ is a proper holomorphic mapping between Riemann surfaces. Let $C \subset X$ be the set of critical points of f, $B = f(C)$ be the set of critical values of f, and $A = f^{-1}(B)$. Set $X' = X - A$ and $Y' = Y - B$. Show that $f : X' \to Y'$ is a proper local homeomorphism and therefore a covering map.

3. Suppose that X is a compact Riemann surface and that $F(T) \in \mathcal{M}(X)$ is an irreducible polynomial. Let

$$\begin{tikzcd} Y \arrow[r, "y"] \arrow[d, "x"'] & \mathbb{P}^1 \\ X & \end{tikzcd}$$

be the Riemann surface of $F(T)$. Show that the natural homomorphism

$$\operatorname{Aut}(Y/X) \to \operatorname{Aut}\big(\mathcal{M}(Y)/\mathcal{M}(X)\big)$$

is an isomorphism.

4. Suppose that k is a field. We can view a polynomial $f(x,y) \in k[x,y]$ as a polynomial $F(y)$ in y whose coefficients lie in $k[x]$. That is, $F(y) \in k[x][y]$. Show that if the coefficients of $F(y)$ are relatively prime in $k[x]$, then $f(x,y)$ is irreducible in $k[x,y]$ if and only $F(y)$ is irreducible in $k(x)[y]$. (Hint: use Gauss' Lemma.) Deduce that if $F(y) \in k(x)[y]$ is irreducible, then there is $a(x) \in k(x)^\times$ such that $f(x,y) := a(x)F(y)$ is irreducible in $k[x,y]$. Show, in this case, that the field

$$k(x)[y]/(F(y))$$

is the field of fractions of the integral domain $k[x,y]/\big(f(x,y)\big)$.

5. Suppose that $f(x,y) \in \mathbb{C}[x,y]$ is irreducible. Denote the corresponding element of $\mathbb{C}(x)[y]$ by $F(y)$. By problem 4, $F(y)$ irreducible. Suppose that the degree of $F(y)$ is $d \geq 1$. Denote the Riemann surface of

$$F(y) \in \mathbb{C}(x)[y] \cong \mathcal{M}(\mathbb{P}^1)[y]$$

by $x : X \to \mathbb{P}^1$.

Let $h(x,y,z)$ be the homogenization of $f(x,y)$:

$$h(x,y,z) := z^e f(x/z, y/z)$$

where e denotes the degree of $f(x,y)$ as a polynomial of two variables. This defines an irreducible plane curve C in $\mathbb{P}^2$ of degree e:

$$C = \{[x,y,z] : h(x,y,z) = 0\}.$$

(i) Let
$$X' = X - \{\text{poles of } x \text{ and } y\}.$$
Define a mapping $\phi : X' \to \mathbb{P}^2$ by
$$\phi(P) = [x(P), y(P), 1].$$

(ii) Show that ϕ extends to a holomorphic mapping[1] $\phi : X \to \mathbb{P}^2$. Hint: clear denominators.

(iii) Show that C is the image of ϕ. Let $C' = C - C^{\text{sing}}$ be the set of points where C is smooth. Set $A = \phi^{-1}(C^{\text{sing}})$. Show that A is finite and that ϕ induces a biholomorphism $\phi : X' \to C'$, where $X' = X - A$. Deduce that $\phi : X \to C$ is an isomorphism when C is smooth.

The mapping $\phi : X \to C$ is an analytic construction of the *normalization* of C.

6. Prove that the group of biholomorphisms of the Riemann sphere $\mathbb{P}^1$ is $\mathrm{PSL}_2(\mathbb{C})$. Show that each such automorphism has a fixed point. Deduce that if X is a Riemann surface whose universal covering is $\mathbb{P}^1$, then X is isomorphic to $\mathbb{P}^1$.

7. Compute $\operatorname{Aut} \mathbb{C}$. Show that if X is a Riemann surface whose universal covering is $\mathbb{C}$, then X is biholomorphic to $\mathbb{C}^*$ or the quotient of $\mathbb{C}$ by a lattice.

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[1] A function $f : X \to \mathbb{P}^2$ is holomorphic if, each $P \in X$ has a neighbourhood U such that $\phi(Q) = [\psi_1(Q), \psi_2(Q), \psi_3(Q)]$, where $\psi_j \in \mathcal{O}(U)$.

8. Find all holomorphic automorphisms of the disk $\mathbb{D}$. (Hint: Use Schwartz's Lemma.) Write down a biholomorphism between $\mathbb{D}$ and the upper half plane $\mathbb{H}$.

9. Show that every automorphism of $\mathbb{H}$ is induced by a fractional linear transformation. Show that a fractional linear transformation $T \in \mathrm{PSL}_2(\mathbb{C})$ induces an automorphism of $\mathbb{H}$ if and only if $T \in \mathrm{PSL}_2(\mathbb{R})$. (Hint: Use the results of the previous problem.) Deduce that $\operatorname{Aut}\mathbb{D} \cong \operatorname{Aut}\mathbb{H} = \mathrm{PSL}_2(\mathbb{R})$.

10. Let X be the Riemann surface of the algebraic function

$$y^2 = \prod_{j=0}^{2g}(x - a_j)$$

where $a_0, \dots, a_{2g}$ are distinct complex numbers.

(i) Show that X has genus g and that the projection $x : X \to \mathbb{P}^1$ is ramified over infinity.
(ii) Show that

$$w_j := x^{j-1}\frac{dx}{y} \qquad 1 \le j \le g$$

are linearly independent holomorphic 1-forms on X. (Hint: the easiest way to do this is to compute the divisor of each w_j.)
(iii) Show that $w_1, \dots, w_g$ span (and therefore comprise a basis of) $\Omega^1(X)$.
(iv) Show that if $p(x)$ is a cubic, then the differential dx/y has no zeros on X.

11. Let $x : X \to \mathbb{P}^1$ by the Riemann surface of the Fermat curve, $x^d + y^d = 1$.

(i) Show that the projection x is ramified only over the dth roots of unity.
(ii) Compute the genus of X.
(iii) Compute $\operatorname{Aut}(X/\mathbb{P}^1)$.