Richard Hain September 1, 2006

Math 272
Riemann Surfaces
Problem Set 1

Due: Tuesday, September 19, 2006.

1. Show that a holomorphic bijection between Riemann surfaces is a biholomorphism.

2. Show that if $f : X \to Y$ is a non-constant holomorphic map between Riemann surfaces, then the inverse image of each $y \in Y$ is discrete in X.

3. Suppose that $p : X \to Y$ is a covering map. Show that if Y is a Riemann surface, then X has a unique complex structure such that p is holomorphic. Note that, in particular, the universal covering of every Riemann surface is naturally a Riemann surface. Show that if Z is a Riemann surface, then a function $f : Z \to X$ is holomorphic if and only if $p \circ f : Z \to Y$ is holomorphic.

4. Suppose that $p : X \to Y$ is a Galois (i.e., normal or regular) covering map with Galois group (i.e., group of deck transformations) G. Suppose that X is a Riemann surface and that G acts on X as a group of biholomorphisms. Show that Y has a unique complex structure such that p is holomorphic.

5. Show that all finite coverings of the punctured disk $\mathbb{D}^*$ are isomorphic to $p_n : \mathbb{D}^* \to \mathbb{D}^*$ where $p_n(z) = z^n$. Deduce that all such coverings $U \to \mathbb{D}^*$ can be completed to a proper holomorphic map $X \to \mathbb{D}$ where X is a Riemann surface containing U as an open subset.

6. Supposed that X is a Riemann surface and that $P \in X$. Let

$$\operatorname{Aut}(X, P) = \{\phi \in \operatorname{Aut} X : \phi(P) = P\}.$$

Show that taking ϕ to its derivative at P defines a surjective homomorphism $\rho : \operatorname{Aut}(X, P) \to \mathbb{C}^*$. Denote its kernel by $\operatorname{Aut}_0(X, P)$.

(i) Show that $\operatorname{Aut}_0(X, P)$ is torsion free. (Hint: use powerseries.)
(ii) Deduce that if G is a finite subgroup of $\operatorname{Aut}(X, P)$ of order d, then the restriction of ρ to G is injective and that $\rho(G)$ is the group of dth roots of unity.
(iii) With G as above, show that there is a holomorphic coordinate z in X, centered at P, such that (for Q in a neighbourhood of

P) the action of G is given by $g : z \mapsto \rho(g)z$. More precisely,

$$z(g(Q)) = \rho(g)z(Q).$$

Hint: Let w be any holomorphic coordinate centered at P and consider how G acts on a dth root of $\prod_{g\in G} g^*w$.

(iv) Show that $G\backslash X$ has a natural Riemann surface structure such that the projection $X \to G\backslash X$ is holomorphic. (Hint: Localize about each fixed point of G.)

7. Let Λ be a lattice in $\mathbb{C}$. Define

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda-\{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right).$$

(i) Show that $\wp_\Lambda$ is a doubly periodic meromorphic function with period lattice Λ. Show that its poles are at the points of Λ. Hint: first prove that its 'formal derivative' exists:

$$\wp'_\Lambda(z) = -2\sum_{\lambda\in\Lambda} \frac{1}{(z-\lambda)^3}.$$

(ii) Deduce that $\wp_\Lambda : \mathbb{C}/\Lambda \to \mathbb{P}^1$ is a $2:1$ holomorphic map which is branched at the four points of order two of $\mathbb{C}/\Lambda$.

(iii) Show that if $f : \mathbb{C} \to \mathbb{P}^1$ is a doubly periodic meromorphic function (with period lattice Λ) with a Laurent expansion of the form

$$f(z) = \sum_{k=-2}^{\infty} c_k z^k$$

with $c_{-2} = 1$ and $c_{-1} = c_0 = 0$ about zero and poles only on Λ, then $f = \wp_\Lambda$.