Richard Hain September 20, 2009

MATH 272
RIEMANN SURFACES
PROBLEM SET 1

Due: Thursday, October 1, 2009.

1. Let Λ be a lattice in $\mathbb{C}$. Define

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda - \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

(i) Show that $\wp_\Lambda$ is a doubly periodic meromorphic function with period lattice Λ. Show that its poles are at the points of Λ. Hint: first prove that its 'formal derivative'

$$\wp'_\Lambda(z) = -2 \sum_{\lambda \in \Lambda} \frac{1}{(z-\lambda)^3},$$

converges almost uniformly on $\mathbb{C} - \Lambda$, then integrate.

(ii) Deduce that $\wp_\Lambda : \mathbb{C}/\Lambda \to \mathbb{P}^1$ is a $2:1$ holomorphic map which is branched at the four points of order two of $\mathbb{C}/\Lambda$. Deduce also that $\wp'_\Lambda$ has only 3 zeros, counting multiplicity.

(iii) Show that if $f : \mathbb{C} \to \mathbb{P}^1$ is a doubly periodic meromorphic function (with period lattice Λ) with a Laurent expansion of the form

$$f(z) = \sum_{k=-2}^{\infty} c_k z^k$$

with $c_{-2} = 1$ and $c_{-1} = c_0 = 0$ about zero and poles only on Λ, then $f = \wp_\Lambda$.

(iv) Show that $\operatorname{Aut}((\mathbb{C}/\Lambda)/\mathbb{P}^1)$ is cyclic of order 2 and is generated by $z \mapsto -z$.

(v) Show that $(\wp'_\Lambda)^2 \in \mathbb{C}[\wp_\Lambda]$. Show that if $(\wp'_\Lambda)^2 = f(\wp_\Lambda)$, then

$$\mathcal{M}(\mathbb{C}/\Lambda) \cong \mathbb{C}(\wp)[T]/(T^2 - f(\wp_\Lambda)).$$

(vi) Use the fact that $\wp'_\Lambda$ has 3 simple zeros to show that $f(T)$ has to be a cubic polynomial with 3 distinct roots. By considering the Laurent expansion about $z = 0$, show that f is of the form

$$f(T) = 4T^3 + aT + b.$$

2. Suppose that X and Y are compact Riemann surfaces and that $A \subset X$ and $B \subset Y$ are finite subsets. Show that every proper holomorphic map $f : X - A \to Y - B$ extends to a holomorphic map $\hat{f} : X \to Y$.

3. Suppose that X and Y are compact Riemann surfaces whose fields of meromorphic functions separate points.[1] Show that X and Y are isomorphic if and only if $\mathcal{M}(X) \cong \mathcal{M}(Y)$.

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[1] That is, if $x, x' \in X$, then there exists $f \in \mathcal{M}(X)$ such that $f(x) \neq f(x')$. Similarly for Y.