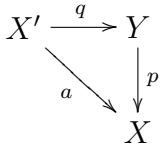
Richard Hain February 15, 2012

Math 261
Algebraic Topology I
Problem Set 3

Due: Thursday, March 1, 2012

1. Suppose that X is a connected topological space. A covering map $p : Y \to X$ is an abelian covering of X if it is a regular (i.e., Galois) covering of X and $\mathrm{Aut}(Y/X)$ is abelian. Show that if X has a universal covering (e.g., X is locally simply connected), then X has an abelian covering $a : X' \to X$ which dominates all other abelian coverings: if $Y \to X$ is an abelian covering, then there is a covering mapping $q : X' \to Y$ such that the diagram



The covering $a : X' \to X$ is called a *maximal abelian covering* of X. Prove that any two maximal abelian coverings of X are isomorphic. (Often we will abuse terminology and refer to *the* maximal abelian covering of X.

2. Suppose that X and Y are connected, locally path connected spaces. Suppose that $p : (Y, y) \to (X, x)$ is a normal covering map. Show that if Z is a path connected subset of X that contains x then the set of path components of $p^{-1}(Z)$ is in natural bijection with the coset space

$$\mathrm{Aut}(Y/X)/\operatorname{im}\pi_1(Z, z)$$

where $\operatorname{im}\pi_1(Z, z)$ denotes the image of the natural homomorphism

$$\pi_1(Z, z) \to \pi_1(X, x) \to \mathrm{Aut}(Y/X).$$

3. Consider the bouquet

$$X = \bigvee_{j=1}^{n} S^1$$

of n circles. Let $j : X \to (S^1)^n$ be the standard inclusion. Show that the homomophism

$$j_* : \pi_1(X, x_o) \to \pi_1\big((S^1)^n, 0\big)$$

is surjective and that $\pi_1\big((S^1)^n, 0\big)$ is the maximal abelian quotient of $\pi_1(X, x_o)$. Deduce that the inverse image Y of X in $\mathbb{R}^n$ under the

covering map $\mathbb{R}^n \to (S^1)^n$ is connected and that the projection $a : Y \to X$ is the maximal abelian covering of X.

$$\begin{array}{ccc} Y & \xrightarrow{J} & \mathbb{R}^n \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[j]{} & (S^1)^n \end{array}$$

Show that there is a homeomorphism

$$Y \cong \bigcup_{k=1}^{n} \mathbb{Z} \times \cdots \times \mathbb{Z} \times \overset{k}{\overset{|}{\mathbb{R}}} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \subset \mathbb{R}^n.$$

Deduce that $\pi_1(Y, y)$ is a countably generated free group.

4. Assume that the topological spaces X and Y are connected and locally path connected. Assume that X has a universal covering $\widetilde{X} \to X$. (E.g., X is locally simply connected.) Suppose that $p : (Y, y) \to (X, x)$ is a pointed covering. Show that

(i) if $q : (Z, z) \to (Y, y)$ is a connected covering with the property that $p \circ q : (Z, z) \to (X, x)$ is normal (aka, regular or Galois), then $\pi_1(Z, z)$ is a subgroup of the kernel of the monodromy homomorphism

$$\pi_1(X, x) \to \operatorname{Aut} p^{-1}(x).$$

(ii) there is a connected normal covering $(W, w) \to (X, x)$ with the property that

$$\pi_1(W, w) = \ker\{\pi_1(X, x) \to \operatorname{Aut} p^{-1}(x)\}.$$

Show that $(W, w) \to (X, x)$ factors through $(Y, y) \to (X, x)$.

(iii) every connected, normal covering $(Z, z) \to (X, y)$ that factors through $(Y, y) \to (X, x)$, must factor through $(W, w) \to (X, x)$.