1. (18 points) Suppose that

\[ \mathbf{a} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix} \in \mathbb{R}^4 \]

(i) (8 points) Compute \( \mathbf{a} \cdot \mathbf{b} \), the lengths of \( \mathbf{a} \) and \( \mathbf{b} \), and the angle \( \theta \) between them.

\[ \mathbf{a} \cdot \mathbf{b} = -3, \quad \|\mathbf{a}\|^2 = \mathbf{a} \cdot \mathbf{a} = 4, \quad \|\mathbf{b}\|^2 = \mathbf{b} \cdot \mathbf{b} = 9. \] So \( \|\mathbf{a}\| = 2 \) and \( \|\mathbf{b}\| = 3 \). If \( \theta \) is the angle between them, then

\[ \cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\| \|\mathbf{b}\|} = -3/(2 \times 3) = -1/2 \]

from which it follows that \( \theta = 2\pi/3 \).

(ii) (7 points) Compute the projection \( \text{proj}_b \mathbf{a} \) of \( \mathbf{a} \) onto \( \mathbf{b} \). Write \( \mathbf{a} \) as the sum of a vector perpendicular to \( \mathbf{b} \) and one parallel to \( \mathbf{b} \).

\[ \text{proj}_b \mathbf{a} = \left( \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} \right) \mathbf{b} = -3/9 \mathbf{b} = (1/3)[-1, -2, 0, 2]^T. \]

Denote this by \( \mathbf{p} \). Then

\[ \mathbf{a} = \mathbf{p} + (\mathbf{a} - \mathbf{p}), \]

where \( \mathbf{p} \) is parallel to \( \mathbf{b} \) and

\[ \mathbf{a} - \mathbf{p} = [1, -1, 1, 1]^T + (1/3)[1, 2, 0, -2]^T = (1/3)[4, -1, 3, 1]^T. \]

is perpendicular to \( \mathbf{b} \).

(iii) (3 points) Is the matrix product \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 & 1 \end{bmatrix} \) defined? If so, compute it.

It is defined because the first matrix is \( 2 \times 1 \) and the second is \( 1 \times 4 \). The answer is thus the \( 2 \times 4 \) matrix

\[ \begin{bmatrix} -1 & 0 & 2 & 1 \\ -2 & 0 & 4 & 2 \end{bmatrix}. \]
2. (30 points) Suppose that
\[
A := \begin{bmatrix}
1 & 0 & 1 & -1 & 0 \\
0 & 1 & 2 & 1 & -1 \\
1 & 1 & 1 & 2 & -1 \\
1 & 2 & 0 & 2 & 0
\end{bmatrix} \in \mathbb{M}_{5 \times 5}(\mathbb{R}).
\]

(i) (12 points) Let \( b = [b_1, b_2, b_3, b_4, b_5]^T \). By reducing \( [A|b] \) to echelon form, find necessary and sufficient conditions on the entries of \( b \) in order that \( Ax = b \) be consistent.

\[
\begin{bmatrix}
1 & 0 & 1 & -1 & 0 & b_1 \\
0 & 1 & 1 & 2 & -1 & b_2 \\
1 & 2 & 0 & 2 & 0 & b_3 \\
1 & 0 & 1 & -1 & 0 & b_4 \\
0 & 1 & -1 & 0 & 1 & b_5
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 1 & -1 & 0 & b_1 \\
0 & 1 & 1 & 2 & -1 & b_2 \\
1 & 2 & 0 & 2 & 0 & b_3 \\
1 & 0 & 1 & -1 & 0 & b_4 \\
0 & 1 & -1 & 0 & 1 & b_5
\end{bmatrix}
\]

So
\[
U = \begin{bmatrix}
1 & 0 & 1 & -1 & 0 \\
0 & 1 & 1 & 2 & -1 \\
0 & 0 & 1 & 3 & -2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

is an echelon form of \( A \). Since the last two rows of \( U \) are zero, the equation \( Ax = b \) is consistent if and only if
\[
b_1 - b_3 + b_4 = 0 \quad \text{and} \quad b_1 - b_2 - b_3 + b_5 = 0.
\]
(ii) (2 points) What is the rank of $A$?

Since $U$ has 3 non-zero rows, the rank of $A$ is 3.

(iii) (8 points) Find the general solution of $Ax = 0$.

The solutions of $Ax = 0$ and $Ux = 0$ are the same. So we solve

$$
\begin{align*}
x_1 + x_3 - x_4 &= 0 \\
x_2 - x_3 + x_5 &= 0 \\
x_3 + 3x_4 - 2x_5 &= 0
\end{align*}
$$

The free variables are $x_4$ and $x_5$. The solution is

$$
x = \begin{bmatrix}
4x_4 - 2x_5 \\
3x_4 + x_5 \\
-3x_4 + 2x_5 \\
x_4 \\
x_5
\end{bmatrix} = x_4 \begin{bmatrix}
4 \\
-3 \\
-3 \\
1 \\
0
\end{bmatrix} + x_5 \begin{bmatrix}
-2 \\
1 \\
2 \\
0 \\
1
\end{bmatrix}
$$

(iv) (8 points) Is $Ax = b$ consistent when $b = [2, 1, 2, 0, 1]^T$? If so, find the general solution of $Ax = b$.

Since

$$b_1-b_2-b_3+b_5 = 2-1-2+1 = 0 \quad \text{and} \quad b_1-b_2-b_3+b_5 = 2-1-2+1 = 0,$$

the system $Ax = b$ is consistent. To find a particular solution, solve

$$
Ux = \begin{bmatrix}
b_1 \\
b_2 \\
-2b_2 - b_2 + b_3 \\
b_1 - b_3 + b_4 \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
2 \\
1 \\
-3 \\
0 \\
0
\end{bmatrix}
$$

with both free variables $x_4$ and $x_5$ set to 0:

$$x_1 + x_3 = 2, \ x_2 - x_3 = 1, \ x_3 = -3.$$

This has solution $x_P = [5, -2, -3, 0, 0]^T$. The general solution is the sum of this and the general solution of the homogeneous equation which you found in (iii):

$$
x = \begin{bmatrix}
5 \\
-2 \\
-3 \\
0 \\
0
\end{bmatrix} + x_4 \begin{bmatrix}
4 \\
-3 \\
-3 \\
1 \\
0
\end{bmatrix} + x_5 \begin{bmatrix}
-2 \\
1 \\
2 \\
0 \\
1
\end{bmatrix}$$
3. (12 points) Is the matrix (with entries in the field \( \mathbb{F}_2 \))

\[
\begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 \\
\end{bmatrix} \in M_{4 \times 4}(\mathbb{F}_2)
\]

non-singular? If so, find its inverse.

Call the matrix \( A \). This is best approached by row reducing \([A|I]\) to the form \([I|B]\) if possible. (You can do this if and only if \( A \) is non-singular.)

\[
[A|I] = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\sim \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
\end{bmatrix}
\]

At this point we see that \( A \) is non-singular, and therefore invertible. Continuing, we have:

\[
[A|I] \sim \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{bmatrix}
\]

\[
\sim \begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{bmatrix}
\]

\[
\sim \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{bmatrix}
\]
The inverse of $A$ is thus:
\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1
\end{bmatrix}
\]

4. (10 points) (a) (4 points) Write down a formula for the inverse (when it exists) of the matrix

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \in \mathbb{M}_{2\times2}(F).
\]

This $2 \times 2$ matrix is invertible if and only if $ad - bc \neq 0$. If $ad - bc \neq 0$, then

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix}
d & -b \\
-c & a
\end{bmatrix}
\]

(b) (6 points) Find the inverse of the matrix

\[
\begin{bmatrix}
i & 1-i \\
1+i & 1
\end{bmatrix} \in \mathbb{M}_{2\times2}(\mathbb{C})
\]

Make sure that each entry of your answer is in the form $a + bi$ where $a$ and $b$ are real numbers.

First,

\[
ad - bc = i - (i + i)(1 - i) = i - 2 \neq 0 \quad \text{and} \quad \frac{1}{ad - bc} = \frac{1}{i - 2} = -(i + 2)/5.
\]

So the inverse exists and is

\[
\begin{bmatrix}
i & 1-i \\
1+i & 1
\end{bmatrix}^{-1} = \frac{2 + i}{5} \begin{bmatrix}1 & -1 + i \\
-1 - i & i
\end{bmatrix} = \frac{1}{5} \begin{bmatrix}-2 - i & 3 - i \\
1 + 3i & 1 - 2i
\end{bmatrix}.
\]

Note: The easiest way to find the inverse of a $2 \times 2$ matrix is to use the formula. For larger matrices, it is easiest to use row reduction.

5. (12 points) Suppose that $P \in \mathbb{M}_{n\times n}(F)$ satisfies $P^2 = P$. Set $Q = I_n - P$.

(i) (6 points) Show that $Q^2 = Q$ and that $PQ = QP = 0$.

This is just a few short calculations:

\[
Q^2 = (I - P)(I - P) = I^2 - 2P + P^2 = I - 2P + P = I - P = Q,
\]

\[
PQ = P(I - P) = P - P^2 = P - P = 0.
\]

\[
QP = (I - P)P = P - P^2 = P - P = 0.
\]
(ii) (6 points) Show that every \( x \in F^n \) can be written uniquely as a sum \( x = x' + x'' \), where

\[
x' = Px', \quad x'' = Qx'' \quad \text{and} \quad Qx' = Px'' = 0.
\]

**Hint:** take \( x' = Px \) and \( x'' = Qx \).

Following the hint, define \( x' = Px \) and \( x'' = Qx \). Then

\[
Px' = P^2x = Px = x', \quad Qx' = QPx = 0,
\]

\[
Px'' = PQx = 0, \quad Qx'' = Q^2x = Qx = x''.
\]

That is, they satisfy the conditions (1). Since \( I = P + Q \),

\[
x = Ix = (P + Q)x = Px + Qx = x' + x''.
\]

Thus every \( x \in F^n \) can be written in the form \( x = x' + x'' \) where the properties (1) are satisfied.

This decomposition is unique, for if \( x = x' + x'' \), where \( x' \) and \( x'' \) satisfy the properties (1), then

\[
Px = P(x' + x'') = Px' + Px'' = x'
\]

and

\[
Qx = Q(x' + x'') = Qx' + Qx'' = x''.
\]

6. (18 points) For \( A \in \mathbb{M}_{2 \times 2}(\mathbb{R}) \) let \( \mu_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be the function defined by \( \mu_A(x) = Ax \). Find \( 2 \times 2 \) real matrices \( R, S \) and \( T \) such that

(i) (4 points) \( \mu_R \) is reflection in the line \( x_1 = x_2 \).

This reflection takes \( (x_1, x_2) \) to \( (x_2, x_1) \). Since

\[
\begin{bmatrix}
x_2 \\
x_1
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

we have

\[
R = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

(ii) (4 points) \( \mu_S \) is reflection in the \( x_1 \)-axis.

This reflection takes \( (x_1, x_2) \) to \( (x_1, -x_2) \). Since

\[
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

we have

\[
S = \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\]
(iii) (4 points) \( \mu_T \) is rotation (counterclockwise) by \( \pi/2 \) about \( \mathbf{0} \).

This rotation takes \((x_1, x_2)\) to \((-x_2, x_1)\). Since

\[
\begin{bmatrix}
-x_2 \\
x_1
\end{bmatrix}
= 
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

we have

\[
T = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\]

(iv) (6 points) Compute \( RS \). What conclusion do you draw?

\[
RS = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
= \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
= T
\]

We conclude that \( \mu_R \circ \mu_S = \mu_T \). In concrete terms, this means that if you first reflect a point across the \( x_1 \)-axis and then across the line \( x_1 = x_2 \), you will have rotated it \( \pi/2 \) radians counterclockwise about the origin.