

Elliptic Curves and Multiple Zeta Numbers

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definition

Definition

Suppose that $r \geq 1$ and that n_1, \dots, n_r are positive integers with $n_r > 1$. The *multiple zeta number* $\zeta(n_1, \dots, n_r)$ is defined by the convergent series

$$\zeta(n_1, \dots, n_r) = \sum_{0 < k_1 < \dots < k_r} \frac{1}{k_1^{n_1} k_2^{n_2} \dots k_r^{n_r}}.$$

The integer r is called the *depth*, and $n_1 + \dots + n_r$ is called *weight* of $\zeta(n_1, \dots, n_r)$.

zeta values

Multiple zeta values of depth 1 are simply values of the Riemann zeta function

$$\zeta(s) = \sum_{k \geq 1} \frac{1}{k^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

at integers > 1 . When n is even, these are just non-zero rational multiples of π^n :

$$\zeta(2m) = -\frac{(2\pi i)^{2m} B_{2m}}{4m(2m-1)!} \in \pi^{2m} \mathbb{Q}^\times$$

where B_{2m} is the m th Bernoulli number:

$$x/(e^x - 1) = \sum_{n \geq 0} B_n x^n / n.$$

zeta values (ctd)

Example

1

$$\zeta(2) = \pi^2/6, \zeta(4) = \pi^4/90, \zeta(6) = \pi^6/945, \dots$$

2

As is well known $\zeta(3)$ is irrational. It would be interesting to know whether $\zeta(2m+1)$ is an irrational multiple of π^{2m+1} .

products of zeta values

If $n, m > 1$, we have:

$$\begin{aligned}\zeta(n)\zeta(m) &= \left(\sum_{k>0} \frac{1}{k^n}\right) \left(\sum_{\ell>0} \frac{1}{\ell^m}\right) \\ &= \left(\sum_{0<k<\ell} + \sum_{0<k=\ell} + \sum_{0<\ell<k}\right) \frac{1}{k^n \ell^m} \\ &= \zeta(n, m) + \zeta(n + m) + \zeta(m, n).\end{aligned}$$

Thus we have:

$$\zeta(n)\zeta(m) = \zeta(n, m) + \zeta(n + m) + \zeta(m, n) \quad n, m > 1$$

a relation of depth 3

Let's try another. If $n_1, n_3 > 1$, then

$$\begin{aligned}\zeta(n_1)\zeta(n_2, n_3) &= \left(\sum_{k_1 > 0} \frac{1}{k_1^{n_1}} \right) \left(\sum_{0 < k_2 < k_3} \frac{1}{k_2^{n_2} k_3^{n_3}} \right) \\ &= \left(\sum_{0 < k_1 < k_2 < k_3} + \sum_{0 < k_1 = k_2 < k_3} + \sum_{0 < k_2 < k_1 < k_3} \right. \\ &\quad \left. + \sum_{0 < k_2 < k_1 = k_3} + \sum_{0 < k_2 < k_3 < k_1} \right) \frac{1}{k_1^{n_1} k_2^{n_2} k_3^{n_3}} \\ &= \zeta(n_1, n_2, n_3) + \zeta(n_1 + n_2, n_3) + \zeta(n_2, n_1, n_3) \\ &\quad + \zeta(n_2, n_1 + n_3) + \zeta(n_2, n_3, n_1).\end{aligned}$$

stuffle relations

The identity

$$\zeta(n_1)\zeta(n_2, n_3) = \zeta(n_1, n_2, n_3) + \zeta(n_1 + n_2, n_3) + \zeta(n_2, n_1, n_3) \\ + \zeta(n_2, n_1 + n_3) + \zeta(n_2, n_3, n_1)$$

is an example of a *stuffle relation*. It implies:

Theorem

The product of two multiple zeta numbers is an integral linear combination of multiple zeta numbers. If the two MZNs have weights w_1 and w_2 , then their product is a sum of MZNs of weight $w_1 + w_2$. If the two MZNs have depths r_1 and r_2 , then their product is a sum of MZNs of depth $\leq r_1 + r_2$.

the algebra MZN_{\bullet} .

The stuffle relations imply that the \mathbb{Q} -subspace of \mathbb{R} spanned by the multiple zeta numbers (MZNs) is an algebra. It is better to think of MZNs as spanning a *graded* algebra. For $n \geq 1$ set

$$MZN_n :=$$

the \mathbb{Q} -subspace of \mathbb{R} spanned by MZNs of weight n .

Let $MZN_0 = \mathbb{Q}$. There is a product $MZN_n \times MZN_m \rightarrow MZN_{m+n}$,
so

$$MZN_{\bullet} := \bigoplus_{n \geq 0} MZN_n$$

is an graded algebra.

a problem

Problem

Compute (or estimate) $\dim_{\mathbb{Q}} \text{MZN}_n$ for all n .

Approaches

- 1 find algebraic relations that hold between MZNs;
- 2 use the techniques of transcendental number theory;
- 3 other techniques?

naive estimate

If all $\zeta(n_1, \dots, n_r)$ of weight n were linearly independent, then the dimension of MZN of weight $n \geq 2$ and depth $r \geq 1$ would be

$$\begin{aligned} &= \# \text{ degree } n \text{ monomials } x_1^{n_1} \dots x_r^{n_r} \quad n_j \geq 1, n_r \geq 2 \\ &= \# \text{ degree } n - r - 1 \text{ monomials } x_1^{m_1} \dots x_r^{m_r} \quad m_j \geq 0 \\ &= \binom{n-2}{r-1} \end{aligned}$$

In this case $\dim_{\mathbb{Q}} \text{MZN}_n$ would be

$$\sum_{r=1}^{n-1} \binom{n-2}{r-1} = \sum_{k=0}^{n-2} \binom{n-2}{k} = 2^{n-2}.$$

a graded algebra

Consider the algebra $A = \mathbb{Q}[Z_2] \otimes \mathbb{Q}\langle Z_3, Z_5, Z_7, \dots \rangle$ where $\mathbb{Q}\langle Z_3, Z_5, Z_7, \dots \rangle$ denotes the free associative algebra generated by indeterminates Z_3, Z_5, Z_7, \dots . If we give Z_m weight m , then this algebra can be graded by weight:

$$A = \bigoplus_{n \geq 0} A_n$$

where A_n is the span of all monomials of weight n .

Example

- 1 A_4 is spanned by Z_2^2 , so $\dim A_4 = 1$
- 2 A_5 is spanned by Z_5 and $Z_2 Z_3$, so $\dim A_5 = 2$

Zagier's Conjecture, Terasoma's Theorem

Conjecture (Zagier, 1994), Theorem (Terasoma, 2003)

For all $n \geq 0$ $\dim \text{MZN}_n \leq \dim A_n$ (N.B. $\dim A_n \sim (0.754878)^n$)

All proofs use the theory of *mixed Tate motives* due to Deligne and Goncharov (which uses work of Voevodsky and Levine).

Example

This implies that the 2^{4-2} MZN of weight 4:

$$\zeta(4), \zeta(2, 2), \zeta(1, 3), \zeta(1, 1, 2)$$

are proportional, and thus all rational multiples of π^4 .

The problem is to find enough relations between MZN of weight n to cut the dimension down to $\dim A_n$.

iterated line integrals

Definition (K.-T. Chen)

Suppose that M is a smooth manifold. (E.g. $\mathbb{C} - \{0, 1\}$.) For 1-forms $\omega_1, \dots, \omega_r$ and a piecewise smooth path $\gamma : [0, 1] \rightarrow M$, define

$$\int_{\gamma} \omega_1 \dots \omega_r = \int_{0 \leq t_1 \leq t_2 \leq \dots \leq t_r \leq 1} \dots \int f_1(t_1) f_2(t_2) \dots f_r(t_r) dt_1 dt_2 \dots dt_r$$

where $\gamma^* \omega_j = f_j(t) dt$.

When $r = 1$, this reduces to the standard line integral $\int_{\gamma} \omega$.

convergence

Take $M = \mathbb{C} - \{0, 1\}$ and let $\omega_0 = \frac{dz}{z}$, $\omega_1 = \frac{dz}{1-z}$. The iterated integral

$$\int_0^1 \omega_{e_1} \omega_{e_2} \dots \omega_{e_r} \quad e_j \in \{0, 1\}$$

converges if and only if $e_1 = 1$ and $e_r = 0$.

example

$$\begin{aligned}\int_0^1 \omega_1 \omega_0 &= \int_0^1 \int_0^z \frac{dw}{1-w} \frac{dz}{z} = \sum_{n=1}^{\infty} \int_0^1 \left(\int_0^z w^{n-1} dw \right) \frac{dz}{z} \\ &= \sum_{n=1}^{\infty} \int_0^1 \frac{z^{n-1}}{n} dz = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2)\end{aligned}$$

Iterating this argument yields:

Formula for $\zeta(n)$

$$\zeta(n) = \int_0^1 \omega_1 \overbrace{\omega_0 \dots \omega_0}^{n-1}$$

Kontsevich's formula

Kontsevich observed that this generalizes:

Kontsevich formula

$$\zeta(n_1, \dots, n_r) = \int_0^1 \omega_1 \overbrace{\omega_0 \dots \omega_0}^{n_1-1} \omega_1 \overbrace{\omega_0 \dots \omega_0}^{n_2-1} \dots \omega_1 \overbrace{\omega_0 \dots \omega_0}^{n_r-1}$$

For example

$$\zeta(2, 1, 3) = \int_0^1 \omega_1 \omega_0 \omega_1 \omega_1 \omega_0 \omega_0$$

Thus every convergent iterated integral of ω_0 and ω_1 over $[0, 1]$ is an MZN.

shuffle product

Iterated integrals satisfy the

shuffle product formula

$$\int_{\gamma} \omega_1 \dots \omega_r \int_{\gamma} \omega_{r+1} \dots \omega_{r+s} = \sum_{\sigma \in \text{sh}(r,s)} \int_{\gamma} \omega_{\sigma(1)} \omega_{\sigma(2)} \dots \omega_{\sigma(r+s)}$$

For example

$$\int_{\gamma} \omega_1 \int_{\gamma} \omega_2 \omega_3 = \int_{\gamma} \omega_1 \omega_2 \omega_3 + \int_{\gamma} \omega_2 \omega_1 \omega_3 + \int_{\gamma} \omega_2 \omega_3 \omega_1.$$

shuffle relations

The shuffle product formula can be used to compute products of MZNs. For example

$$\begin{aligned}\zeta(2)^2 &= \int_0^1 \omega_1 \omega_0 \int_0^1 \omega_1 \omega_0 \\ &= 2 \int_0^1 \omega_1 \omega_0 \omega_1 \omega_0 + 4 \int_0^1 \omega_1 \omega_1 \omega_0 \omega_0 \\ &= 2\zeta(2, 2) + 4\zeta(1, 3).\end{aligned}$$

This is an example of a *shuffle relation*.

double shuffle relations

New relations can be obtained by equating the stuffle and shuffle formulas for the product of two MZNs. For example, equating the stuffle formula

$$\zeta(2)^2 = 2\zeta(2, 2) + \zeta(4)$$

with the shuffle formula

$$\zeta(2)^2 = 2\zeta(2, 2) + 4\zeta(1, 3)$$

we see that

$$4\zeta(1, 3) = \zeta(4).$$

This is an example of a *double shuffle relation* — cf. Racinet.

further properties of iterated integrals

Iterated integrals satisfy other useful relations, such as:

antipode and naturality properties

$$\textcircled{1} \int_{\gamma^{-1}} \omega_1 \dots \omega_r = (-1)^r \int_{\gamma} \omega_r \dots \omega_1$$

$$\textcircled{2} \int_{f \circ \gamma} \omega_1 \dots \omega_r = \int_{\gamma} f^* \omega_1 \dots f^* \omega_r$$

where $\gamma : [0, 1] \rightarrow N$, $f : N \rightarrow M$ and $\omega_1, \dots, \omega_r$ are 1-forms on M .

inversion formula

Applied to the automorphism $f : z \mapsto 1 - z$ of $\mathbb{C} - \{0, 1\}$, this gives

$$\int_0^1 \omega_{e_1} \dots \omega_{e_r} = \int_0^1 \omega_{1-e_r} \dots \omega_{1-e_1}$$

where each $e_j \in \{0, 1\}$ as $f^* \omega_e = -\omega_{1-e}$.

For example,

$$\zeta(3) = \int_0^1 \omega_1 \omega_0 \omega_0 = \int_0^1 \omega_1 \omega_1 \omega_0 = \zeta(1, 2).$$

and

$$\zeta(4) = \int_0^1 \omega_1 \omega_0 \omega_0 \omega_0 = \int_0^1 \omega_1 \omega_1 \omega_1 \omega_0 = \zeta(1, 1, 2).$$

back to Zagier's Conjecture

Zagier's Conjecture predicts that $\dim \text{MZN}_3 = 1$ and $\dim \text{MZN}_4 = 1$. We have just seen that the inversion formula implies that $\zeta(3) = \zeta(1, 2)$, which proves Zagier's conjecture in weight 3. In weight 4, we have

$$\zeta(2)^2 = \left(\frac{\pi^2}{6}\right)^2 = \frac{5}{2} \frac{\pi^4}{90} = \frac{5}{2} \zeta(4) \quad \text{classical}$$

$$\zeta(2, 2) = \frac{1}{2} (\zeta(2)^2 - \zeta(4)) = \frac{3}{4} \zeta(4) \quad \text{stuffle}$$

$$\zeta(1, 3) = \frac{1}{4} \zeta(4) \quad \text{double shuffle}$$

$$\zeta(1, 1, 2) = \zeta(4) \quad \text{inversion}$$

So our identities imply Zagier's Conjecture in weight 4 as well.

Elementary relations, such as Euler, stuffle, shuffle and inversion, do not always imply Zagier's bound. The first interesting cases occur in weights 12 and 16.

Theorem (Gangl-Kaneko-Zagier, 2006)

$$\begin{aligned} 28 \zeta(3, 9) + 150 \zeta(5, 7) + 168 \zeta(7, 5) &= \frac{5197}{691} \zeta(12) \\ 66 \zeta(3, 13) + 375 \zeta(5, 11) + 686 \zeta(7, 9) \\ &+ 396 \zeta(11, 5) = \frac{78967}{3617} \zeta(16) \end{aligned}$$

These relation comes from the *cuspidal forms of weight 12 and 16* of the *modular group* $SL_2(\mathbb{Z})$. This relation generalizes to all cuspidal forms of $SL_2(\mathbb{Z})$.

modular group

The modular group

$$SL_2(\mathbb{Z}) = \{2 \times 2 \text{ integer matrices of determinant } 1\}$$

acts on the upper half plane $\mathfrak{h} := \{\tau \in \mathbb{C} : \text{im } \tau > 0\}$ by fractional linear transformations.

The quotient $SL_2(\mathbb{Z}) \backslash \mathfrak{h}$ is the “moduli space $\mathcal{M}_{1,1}$ of elliptic curves”.

modular forms

Definition

A *modular form of weight n* is a holomorphic function $f : \mathfrak{h} \rightarrow \mathbb{C}$ that satisfies

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^n f(\tau) \quad A \in \mathrm{SL}_2(\mathbb{Z})$$

(in particular $f(\tau + 1) = f(\tau)$) and its Fourier expansion

$$f(\tau) = \sum_{n \in \mathbb{Z}} a_n q^n \quad q = e^{2\pi i \tau}$$

satisfies $a_n = 0$ when $n < 0$. It is a *cuspidal form* if, in addition, $a_0 = 0$.

examples

There are no non-zero modular forms of odd weight.

Examples

- ① The *Eisenstein series* G_{2k} of weight $2k \geq 4$:

$$G_{2k}(\tau) = \sum_{(m,n) \neq (0,0)} \frac{1}{(m\tau + n)^{2k}}.$$

- ② The *Ramanujan tau function*

$$\Delta(\tau) = (G_2/60)^3 - (G_3/140)^2 = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

is a cusp form of weight 12.

modular symbols

The coefficients of the relation between MZNs that comes from the cusp form f of weight $2n$ are derived from the *modular symbol* of f , which is defined to be the homogeneous polynomial

$$r_f(X, Y) := \int_0^{i\infty} f(\tau)(\tau Y - X)^{2n-2} d\tau \in \mathbb{C}[X, Y]$$

of degree $2n - 2$.

Why modular symbols give relations between MZN remains mysterious.

cuspidal forms and relations between MZV

Why should cuspidal forms give relations between MZV?

- MZV are “periods” of *mixed Tate motives*;
- there is a theory of *mixed elliptic motives* — Hain-Matsumoto (Goncharov);
- mixed elliptic motives become mixed Tate motives when restricted to the nodal cubic $y^2 = x^2(x - 1)$;
- cuspidal forms give relations between mixed elliptic motives — H-M.
- this is explicit in certain cases (Pollack) — and Pollack's relations imply the Galois analogue (Ihara) of these relations between MZV.