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The Geometry of the Mixed Hodge Structure on the Fundamental Group

RICHARD M. HAIN

In Hodge theory, one studies the moduli of a variety V by putting a Hodge structure on its integral cohomology groups $H^m(V;\mathbf{Z})$ and examining how the Hodge filtration moves when the complex structure of V is deformed. In concrete terms, one studies the periods of the differentials on V with a prescribed number of dz's. Suppose now that one wants to study the moduli of a pointed variety[1] (V,x) using Hodge theory. This may be interesting, for example, when x is a variety viewed as a point in a specific moduli space V. One first needs a topological invariant of (V,x). The first that comes to mind is its fundamental group, $\pi_1(V,x)$. Since this is not an abelian group, it is better to replace it by its integral group ring $\mathbf{Z}\pi_1(V,x)$ which is. Since this has infinite rank when π_1 is infinite, and for reasons that will soon become apparent, we replace it by a truncation

$$\mathbf{Z}\pi_1(V,x)/J^{s+1}$$

by some power of its augmentation ideal[2] J. This is a finitely generated abelian group. Next one needs a Hodge filtration.

In the classical case, the position of the Hodge filtration inside $H^{\cdot}(V;\mathbf{C})$ is given by periods of integrals over integral cycles. To do the same for homotopy, one needs a de Rham theory for the fundamental group. K.-T. Chen has developed such a de Rham theory using iterated integrals [**5**]. An iterated integral is a linear combination of basic iterated integrals, denoted $\int w_1 w_2 \cdots w_r$, where each w_j is a 1-form on V. Each defines a function

$$\{\text{loops in } V \text{ based at } x\} \to \mathbf{C}.$$

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[1]That is, a variety V and a point x of V.

[2]Recall that the augmentation ideal is the kernel of the ring homomorphism $\mathbf{Z}\pi_1(V,x) \to \mathbf{Z}$ that takes each element of π_1 to 1.

The collection of iterated integrals of length $\leq s$ whose value on each loop based at x depends only on its homotopy class will be denoted by $H^0(B_s(V),x)$. Each element of $H^0(B_s(V),x)$ defines a $\mathbf{Z}$-linear map

$$\mathbf{Z}\pi_1(V,x)/J^{s+1} \to \mathbf{C}$$

via integration. Chen's π_1 de Rham theorem asserts that the integration map

$$H^0(B_s(V),x) \to \mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}\pi_1(V,x)/J^{s+1},\mathbf{C})$$

is an isomorphism for each $s \geq 0$. Thus the appropriate analogues of the integral cohomology groups for a pointed variety (V,x) are the groups

$$\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}\pi_1(V,x)/J^{s+1},\mathbf{Z}).$$

A Hodge filtration on $H^0(B_s(V),x)$ can be defined in the obvious way. Namely, an iterated integral $\int w_1 \cdots w_r$ is in F^p if the total number of dz's in the w_j's is $\geq p$. The weight is given by the length filtration when V is smooth and projective:

$$W_l(\mathbf{Z}\pi_1(V,x)/J^{s+1})^* = (\mathbf{Z}\pi_1(V,x)/J^{l+1})^* \cong H^0(B_l(V),x).$$

This is clearly defined over $\mathbf{Q}$. If $H^1(V)$ is a pure Hodge structure of weight 2 (for example, V is a Zariski open subset of $\mathbf{P}^n$), then the weight filtration on $\mathbf{Z}\pi_1(V,x)/J^{s+1}$ is defined by

$$W_{2l+1} = W_{2l} = (\mathbf{Z}\pi_1(V,x)/J^{l+1})^* \cong H^0(B_l(V),x).$$

In general, the weight filtration on $H^0(B_s(V),x)$ is the convolution of the length filtration of $H^0(B_s(V),x)$ with the weight filtration on the de Rham complex of V. Together the Hodge and weight filtrations on $H^0(B_s(V),x)$ define a mixed Hodge structure (M.H.S.).

Our efforts would have been wasted if the M.H.S. on $\mathbf{Z}\pi_1(V,x)/J^{s+1}$ did not depend upon x. First, when $s=1$ there is an isomorphism of M.H.S.'s

$$\mathbf{C}\pi_1(V,x)/J^2 \cong \mathbf{C} \oplus H_1(V;\mathbf{C}).$$

The M.H.S. on $H_1(V;\mathbf{C})$ is independent of the basepoint. Thus the first possibly interesting case is when $s=2$. Provided that the cup product

$$\Lambda^2 H^1(V;\mathbf{Q}) \to H^2(V;\mathbf{Q})$$

is not injective, the M.H.S. on $\mathbf{Z}\pi_1(V,x)/J^3$ will vary with the basepoint. Sometimes this M.H.S. is very good at finding the basepoint.

(7.5) THEOREM (HAIN, PULTE). *Suppose that (V,x) and (W,y) are two pointed smooth projective curves. If there is a ring isomorphism*

$$\theta\colon \mathbf{Z}\pi_1(V,x)/J^3 \to \mathbf{Z}\pi_1(W,y)/J^3$$

that induces an isomorphism of M.H.S.'s, then there is an isomorphism $f\colon V \to W$ such that, with the possible exception of at most two points x of V, $f(x) = y$. □

When V is a smooth projective curve, there is a deeper connection between the M.H.S. on $\mathbf{Z}\pi_1(V,x)/J^3$ and geometry. Associated with (V,x) is the Abel-Jacobi mapping

$$\nu_x \colon V \to \mathrm{Jac}(V),$$
$$y \to \int_x^y .$$

The image of ν_x is an algebraic 1-cycle V_x in $\mathrm{Jac}(V)$. For a subset A of $\mathrm{Jac}(V)$, we set $A^- = \{-a \colon a \in A\}$. The cycles $V_x - V_y$ and $V_x - V_y^-$ are homologous to zero in $\mathrm{Jac}(V)$ and consequently define points in the intermediate Jacobian

$$J_2(\mathrm{Jac}) = \frac{\mathrm{Hom}(F^2H^3(\mathrm{Jac}), \mathbf{C})}{H_3(\mathrm{Jac}; \mathbf{Z})}.$$

Using techniques of B. Harris [**15, 16**], Pulte [**25**] has shown that the M.H.S.'s on $\mathbf{Z}\pi_1(V,z)/J^3$ $(z = x, y)$ explicitly determine the points in $J_2(\mathrm{Jac})$ determined by $V_x - V_y$ and $V_x - V_y^-$.

The mixed Hodge structure on π_1 also arises naturally when considering monodromy representations. Suppose that $V = X - D$, where X is a smooth projective variety and D is a divisor in X with normal crossings. A $\mathrm{gl}(n)$-valued, holomorphic 1-form on V with logarithmic singularities at infinity is an element ω of $\Omega^1(X \log D) \otimes \mathrm{gl}(n)$. It defines a pfaffian system on V: $df = f\omega$, where $f \colon V \to \mathbf{C}^n$ is a locally defined function. This system is completely integrable if and only if $d\omega + \omega \wedge \omega = 0$.

Analytically continuing the fundamental solution of such a completely integrable system defines a monodromy representation $\rho \colon \pi_1(V,x) \to \mathrm{GL}(n)$. The M.H.S. on $\pi_1(V,x)$ controls the monodromy representation. As an example, we state the following theorem that characterizes unipotent monodromy representations that arise as monodromy representations of nilpotent 1-forms (i.e., 1-forms taking values in a nilpotent subalgebra of $\mathrm{gl}(n)$.

(9.7) THEOREM. *A unipotent representation $\rho \colon \pi_1(V,x) \to \mathrm{GL}(n)$ is the monodromy representation of a completely integrable nilpotent 1-form on V with logarithmic singularities at infinity if and only if there is an algebra homomorphism*

$$\rho \colon [\mathbf{C}\pi_1(V,x)/J^n]/I \to \mathrm{gl}(n)$$

such that the diagram

$$\begin{array}{ccc} \pi_1(V,x) & \to & [\mathbf{C}\pi_1(V,x)/J^n]/I \\ {\scriptstyle \bar\rho}\downarrow & & \downarrow{\scriptstyle \bar\rho} \\ \mathrm{GL}(n) & \underset{\text{inclusion}}{\hookrightarrow} & \mathrm{gl}(n) \end{array}$$

commutes, where I is the ideal

$$F^0 \cap J + F^{-1} \cap J^2 + F^{-2} \cap J^3 + \cdots$$

of $\mathbf{C}\pi_1(V,x)/J^n$. □

In §9 we state the main result of [**30**] which characterizes monodromy representations of integrable, logarithmic 1-forms.

Morgan [**20**] was the first to put a M.H.S. on the Malcev Lie algebra $\mathfrak{g}(V,x)$ of $\pi_1(V,x)$ which he did using Sullivan's minimal models. This M.H.S. determines and is determined by the M.H.S. on the J-adic completion $\mathbf{Q}\pi_1(V,x)^\wedge$ of $\mathbf{Q}\pi_1(V,x)$. ($\mathbf{Q}\pi_1^\wedge$ is the completion of the universal enveloping algebra of $\mathfrak{g}$, while $\mathfrak{g}$ is the set of primitive elements of $\mathbf{Q}\pi_1^\wedge$.) Morgan's construction proceeds by first putting a (not necessarily unique) M.H.S. on the Sullivan minimal model of V. Passing to indecomposables, one gets a family of M.H.S.'s on $\mathfrak{g}(V,x)$. This construction is best suited to topological applications. For example, Morgan has shown that not every finitely presented group can occur as the fundamental group of a smooth variety. But for geometrical applications, the direct construction of the M.H.S. on π_1 using iterated integrals seems more appropriate.

This paper begins with a fairly complete and self-contained account of Chen's de Rham theory for π_1. Iterated line integrals are defined in §1, and their relationship to connections on trivial bundles is explored and exploited in §2. In §3 lots of iterated integrals are constructed that are homotopy functionals and the π_1 de Rham theorem is proved in §4.

The study of the M.H.S. on $\pi_1(V,x)$ begins in §5 where we sketch a proof of its existence. The extension theory for separated extensions of M.H.S. with two nontrivial weights is reviewed in §6 and then applied to the M.H.S. on $J(V,x)/J^3$. General conditions under which the local Torelli theorem is true for the M.H.S. on $J(V,x)/J^3$ are derived in §7 and these are used to prove the almost global Torelli theorem for pointed smooth projective curves.

Pulte's interpretation of B. Harris's harmonic volume is sketched in §8. The relationship between the M.H.S. on $J(V,x)/J^3$ of a smooth projective curve and certain algebraic 1-cycles in its Jacobian is made explicit. Various results announcing new, nontrivial restrictions on the monodromy representations of completely integrable 1-forms with logarithmic singularities are stated in §9.

Most of the ideas in §§1–4 have been harvested from Chen's papers. However, the elementary proof of the π_1 de Rham theorem given in §4 is new.

We assume throughout that the reader is familiar with Deligne's mixed Hodge theory. Two good introductions to it are the papers of Griffiths and Schmid [**10**] and Durfee [**8**]. The accessible paper [**3**] by Carlson is a recommended companion while reading §§6 and 7. An intuitive explanation of how iterated line integrals work is given in the introduction of [**12**].

Finally I would like to thank everyone who has taken interest in this work, particularly Jim Carlson, P. Cartier, P. Deligne, Alan Durfee, Herb Clemens, and Steven Zucker.

1. Iterated integrals. Let M be a smooth manifold. Denote the set of piecewise smooth paths $\gamma\colon [0,1] \to M$ by PM. We will call a function $F\colon PM \to \mathbf{R}^n$ a *homotopy functional* if $F(\gamma)$ depends only on the homotopy class of γ relative to its endpoints. For each $x \in M$, a homotopy functional F induces a function $\pi_1(M,x) \to \mathbf{R}^n$.

Suppose that w is a 1-form on M. The usual line integral

$$\int w: PM \to \mathbf{R},$$
$$\gamma \to \int_\gamma w$$

can be defined as follows:

$$\int_\gamma w = \int_0^1 f(t)\,dt,$$

where $\gamma^* w = f(t)\,dt$ is the pullback of w to $[0,1]$. Now $\int w$ is a homotopy functional if and only if w is closed. Thus the usual line integral can only detect elements of $\pi_1(M,x)$ visible in $H_1(M;\mathbf{R})$.

K.-T. Chen has discovered a generalization of the line integral that often detects many elements of $\pi_1(M)$ that are trivial in homology.

(1.1) DEFINITION. Suppose that $w_1, \ldots, w_r$ are smooth 1-forms on M and that $\gamma \in PM$. Define

$$\int_\gamma w_1 w_2 \cdots w_r = \int \cdots \int_{0 \le t_1 \le \cdots \le t_r \le 1} f_1(t_1) f_2(t_2) \cdots f_r(t_r)\, dt_1 \cdots dt_r,$$

where $f_j(t)\,dt = \gamma^* w_j$. Denote the function $PM \to \mathbf{R}$, $\gamma \to \int_\gamma w_1 \cdots w_r$ by $\int w_1 \cdots w_r$. A linear combination of such functions and the constant function is called an *iterated line integral.* A functional $PM \to \mathbf{R}$ is an iterated line integral of length $\le s$ if it is a linear combination of a constant function and iterated integrals $\int w_1 \cdots w_r$ with $r \le s$.

The definition given above works equally for 1-forms on M that take values in an associative algebra A. In this case

$$\int w_1 \cdots w_r : PM \to A.$$

Two cases of special interest are when $A = \mathbf{C}$ and $A = \mathrm{gl}(n, \mathscr{k})$, the algebra of $n \times n$ matrices over ($\mathscr{k} = \mathbf{R}$ or $\mathbf{C}$).

We will denote the usual de Rham complex of C^∞ forms on M by $E^\cdot(M)$.

The following naturality property of iterated line integrals is easily verified.

(1.2) PROPOSITION. *Suppose that M and N are smooth manifolds, that $\gamma \in PN$, and that $w_1, \ldots, w_r \in E^1(M)$. If $f: N \to M$ is a smooth map, then*

$$\int_\gamma f^* w_1 f^* w_2 \cdots f^* w_r = \int_{f \circ \gamma} w_1 \cdots w_r.$$

The reader should be aware that two iterated integrals that look different may indeed be equal. The following assertion is easily verified.

(1.3) PROPOSITION. *Suppose that* $w_1, \ldots, w_r \in E^1(M)$ *and that* $\gamma \in PM$. *If* $f \in E^0(M)$, *then*

$$\text{(a)} \qquad \int_\gamma df w_1 \cdots w_r = \int_\gamma (f w_1) w_2 \cdots w_r - f \circ \gamma(0) \int_\gamma w_1 \cdots w_r,$$

$$\text{(b)} \qquad \begin{aligned} &\int_\gamma w_1 \cdots w_{i-1}\, df w_i \cdots w_r \\ &\quad = \int_\gamma w_1 \cdots w_{i-1} (f w_i) w_{i+1} \cdots w_r - \int_\gamma w_1 \cdots (f w_{i-1}) w_i \cdots w_r, \end{aligned}$$

$$\text{(c)} \qquad \int_\gamma w_1 \cdots w_r\, df = f \circ \gamma(1) \int_\gamma w_1 \cdots w_r - \int_\gamma w_1 \cdots w_{r-1}(f w_r). \qquad \square$$

Finally, we remark that some interesting functions, such as the higher logarithms, can be written as iterated integrals. Let

$$w_1 = \frac{1}{2\pi\sqrt{-1}} \frac{dz}{1-z}, \qquad w_2 = \frac{1}{2\pi\sqrt{-1}} \frac{dz}{z}.$$

These are rational forms on $\mathbf{P}^1$. The kth higher logarithm is given by the formula

$$\ln_k(z) = \int_0^z w_1 \overbrace{w_2 \cdots w_2}^{k-1}.$$

2. Connections on trivial bundles. In this section we relate iterated integrals to geometry via the transport of a connection on a trivial bundle. This allows us to derive several formal properties of iterated integrals in this section, such as their behaviour on a product of paths. This relationship allows us to find all iterated integrals of length 2 that are homotopy functionals in the next section and to relate the M.H.S. on π_1 to the Riemann-Hilbert problem in §9.

Suppose that ∇ is a connection on the trivial vector bundle $\mathbf{R}^n \times M \to M$. Sections of this bundle correspond to functions $x\colon M \to \mathbf{R}^n$. The canonical framing of this bundle is given by the n constant functions

$$e_i\colon M \to \mathbf{R}^n, \qquad i = 1, \ldots, n,$$

which take each point of M to the ith standard basis vector of $\mathbf{R}^n$. Define the connection form[3] $\omega \in E^1(M) \otimes \mathrm{gl}(n)$ by

$$\nabla e_i = -\sum w_{ij} e_j,$$

where $w_{ij} \in E^1(M)$ and $\omega = (w_{ij})$. Viewing a section of this bundle as a function $x\colon M \to \mathbf{R}^n$, we have

$$(2.1) \qquad \nabla x = dx - x\omega.$$

Conversely, if $\omega \in E^1(M) \otimes \mathrm{gl}(n)$, we can define a connection on $\mathbf{R}^n \times M \to M$ by (2.1). The connection lifts to the bundle $\mathrm{GL}(n) \times M \to M$ by defining $\nabla X = dX - X\omega$, where $X\colon M \to \mathrm{GL}(n)$.

[3] This is not the usual way to define the connection form. We chose this definition so that the transport function would satisfy $T(\alpha \cdot \beta) = T(\alpha)T(\beta)$.

Since the bundle $\mathrm{GL}(n) \times M \to M$ is trivial, sections of it along a smooth map $f\colon N \to M$ correspond to functions $X\colon N \to \mathrm{GL}(n)$. In particular, a section $X\colon [a,b] \to \mathrm{GL}(n)$, along a smooth path $\gamma\colon [a,b] \to M$, is horizontal if and only if

$$dX(t) = X(t)\gamma^*\omega \tag{2.2}$$

for all t. If $\gamma^*\omega = A(t)\,dt$, then (2.2) becomes

$$X'(t) = X(t)A(t). \tag{2.3}$$

Define the *transport function* $T\colon PM \to \mathrm{GL}(n)$ as follows. If $\gamma\colon [0,1] \to M$ is smooth, define $T(\gamma)$ to be $X(1)$, where $X\colon [0,1] \to \mathrm{GL}(n)$ is the unique horizontal section along γ (i.e., solution of (2.3)) satisfying $X(0) = I$. If γ is piecewise smooth, $T(\gamma)$ is defined in the obvious way.

The following properties of the transport function are immediate consequences of the theory of linear ordinary differential equations.

(2.4) PROPOSITION. (a) *If $\gamma \in PM$, then $T(\gamma)$ is independent of the parametrization of γ.*

(b) *If $\alpha, \beta \in PM$ and $\alpha(1) = \beta(0)$, then $T(\alpha\beta) = T(\alpha)T(\beta)$.*

We now give a formula for T in terms of iterated integrals of the connection form. Recall that the definition of iterated integral applies equally well to $\mathfrak{gl}(n)$-valued 1-forms.

Suppose that ∇ is a connection on a trivial vector bundle $\mathbf{R}^n \times M \to M$ with connection form ω and transport function T.

(2.5) LEMMA. (a) *If $\gamma \in PM$, then there exists a constant $M > 0$ such that*

$$\left\| \int_\gamma \overbrace{\omega\omega\cdots\omega}^{r} \right\| = O\left(\frac{M^r}{r!}\right)$$

so that the series

$$I + \int_\gamma \omega + \int_\gamma \omega\omega + \int_\gamma \omega\omega\omega + \cdots$$

converges.

(b)

$$T(\gamma) = I + \int_\gamma \omega + \int_\gamma \omega\omega + \int_\gamma \omega\omega\omega + \cdots.$$

PROOF. Since $T(\gamma)$ is independent of the parametrization of γ, we may assume, by reparametrizing γ if necessary, that γ is smooth. If $\gamma^*\omega = A(t)\,dt$, then $T(\gamma) = X(1)$, where $X(t)$ is the solution of

$$X'(t) = X(t)A(t), \qquad X(0) = I. \tag{$*$}$$

Note that

$$\int_\gamma \overbrace{\omega\omega\cdots\omega}^{r} = \int\cdots\int_{0\le t_1\le\cdots\le t_r\le 1} A(t_1)A(t_2)\cdots A(t_r)\,dt.$$

We will solve (*) by Picard iteration (cf. [**21**]): $X(t)$ satisfies (*) if and only if

$$X(t) = I + \int_0^t X(s)A(s)\,ds.$$

The sequence $\{X_r(t)\}$ of Picard iterates is defined as follows:

$$X_0(t) \equiv I, \qquad X_r(t) = I + \int_0^t X_{r-1}(s)A(s)\,ds.$$

As is well known (cf. [**21**]), there exists $M > 0$ such that

$$\|X_r(t) - X_{r-1}(t)\| = O\left(\frac{M^r}{r!}t^r\right)$$

and $X_r(t)$ converges almost uniformly to $X(t)$.

First we have $X_0(t) = I$ and

$$X_1(t) - X_0(t) = \int_0^t A(s)\,ds.$$

Assume, by induction, that

$$X_n(t) - X_{n-1}(t) = \int\cdots\int_{0\le s_1\le\cdots\le s_n\le t} A(s_1)\cdots A(s_n)\,ds_1\cdots ds_n$$

whenever $n < r$. Now

$$\begin{aligned} X_r(t) - X_{r-1}(t) &= \int_0^t (X_{r-1}(s) - X_{r-2}(s))A(s)\,ds \\ &= \int_0^t \int\cdots\int_{0\le s_1\le\cdots\le s_{r-1}\le s} A(s_1)\cdots A(s_{r-1})A(s)\,ds_1\cdots ds_{r-1}\,ds \\ &= \int\cdots\int_{0\le s_1\le\cdots\le s_r\le t} A(s_1)\cdots A(s_r)\,ds_1\cdots ds_r. \end{aligned}$$

Consequently,

$$X_r(1) - X_{r-1}(1) = \int_\gamma \overbrace{\omega\omega\cdots\omega}^{r}$$

and

$$T(\gamma) = X(1) = \lim_{r\to\infty} X_r(1) = I + \int_\gamma \omega + \int_\gamma \omega\omega + \cdots. \quad \square$$

Of special importance is the case when $\omega \in E^1(M)\otimes\mathfrak{n}_r$, where $\mathfrak{n}_r$ denotes the Lie algebra of nilpotent upper triangular $(r+1)\times(r+1)$ matrices:

$$\mathfrak{n}_r = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}.$$

In this case

$$T = I + \int\omega + \int\omega\omega + \cdots + \int\overbrace{\omega\cdots\omega}^{r}.$$

For example, if $w_1, w_2, \ldots, w_r \in E^1(M)$ and

$$\omega = \begin{pmatrix} 0 & w_1 & 0 & 0 & \cdots & 0 \\ & 0 & w_2 & 0 & \cdots & 0 \\ & & 0 & & & \vdots \\ & 0 & & & & w_r \\ & & & & & 0 \end{pmatrix}, \tag{2.6}$$

then

$$T = \begin{pmatrix} 1 & \int w_1 & \int w_1 w_2 & \int w_1 w_2 w_3 & \cdots & & \int w_1 \cdots w_r \\ 0 & 1 & \int w_2 & \int w_2 w_3 & \cdots & & \int w_2 \cdots w_r \\ 0 & 0 & 1 & \int w_3 & & & \\ & & & 1 & & & \vdots \\ & & & & \ddots & & \\ & 0 & & & & 1 & \int w_r \\ & & & & & & 1 \end{pmatrix}. \tag{2.7}$$

That is,

$$T_{ij} = \begin{cases} \int w_i \cdots w_{j-1}, & j > i, \\ 1, & j = i, \\ 0, & j < i. \end{cases}$$

Since $T(\gamma)$ is independent of the parametrization of γ, (2.4), each entry of $T(\gamma)$ is independent of the parametrization of γ. Applying this to (2.7) we obtain the following generalization of a well-known property of line integrals.

(2.8) PROPOSITION. *If $w_1, \ldots, w_r \in E^1(M)$ and $\gamma \in PM$, then $\int_\gamma w_1 \cdots w_r$ is independent of the parametrization of γ.*

Applying (2.4)(b) to the transport function (2.7) we obtain the following property of iterated integrals.

(2.9) PROPOSITION. *Suppose that $w_1, \ldots, w_r \in E^1(M)$. If $\alpha, \beta \in PM$ and $\alpha(1) = \beta(0)$, then*

$$\int_{\alpha\beta} w_1 \cdots w_r = \sum_{i=0}^{r} \int_\alpha w_1 \cdots w_i \int_\beta w_{i+1} \cdots w_r.$$

Here, we introduce the convention that $\int_\gamma \varphi_1 \cdots \varphi_s = 1$ when $s = 0$. □

Recall that an iterated integral of length $\leq s$ is a finite linear combination

$$I = \lambda + \sum a_i \int w_i + \sum a_{ij} \int w_i w_j + \cdots + \sum_{|J|=s} a_J \int w_{j_1} \cdots w_{j_s}.$$

We shall denote the value of I on the path γ by $\langle I, \gamma \rangle$ or $I(\gamma)$. The pairing $\langle\ ,\ \rangle$ can be extended, by linearity, to a pairing between iterated integrals and 1-chains.

Denote the vector space of iterated integrals on M of length $\leq s$ by $B_s(M)$. Denote the constant path at the point x of M by η_x. (That is, $\eta_x(t) = x$ for all t.) If $r \geq 1$, then

$$\left\langle \int w_1 \cdots w_r, \eta_x \right\rangle = 0$$

for all $x \in M$. Thus, evaluating at a constant path η_x defines a linear functional

$$\varepsilon \colon B_s(M) \to R, \qquad I \to \langle I, \eta_x \rangle$$

that is independent of x. If

$$I = \lambda + \sum a_i \int w_i + \sum a_{ij} \int w_i w_j + \cdots,$$

then $\varepsilon(I) = \lambda$. Denote the kernel of ε by $\overline{B}_\varepsilon(M)$. These are the iterated integrals of length $\leq s$ with zero constant term. Since there is a natural inclusion $i \colon R \to B_s(M)$ such that $\varepsilon \circ i = \mathrm{id}$, we have a natural direct sum decomposition

$$B_s(M) \cong R \oplus \overline{B}_s(M).$$

For loops $\alpha, \beta \in PM$ based at x, we can form the commutator $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. Often we will denote the constant loop η_x at x by 1. Since the paths $\alpha\eta_x$ and $\eta_x\alpha$ differ from the loop α by a reparametrization, it follows from (2.8) that if I is an iterated integral, then

$$I(\alpha) = I(\alpha\eta_x) = I(\eta_x\alpha).$$

Recall that the classical line integral satisfies

$$\left\langle \int w, [\alpha, \beta] \right\rangle = 0 \quad \text{and} \quad \left\langle \int w, (\alpha - 1)(\beta - 1) \right\rangle = 0,$$

where α and β are loops based at x. The following lemma is a generalization of this fact.

(2.10) LEMMA. *Suppose that $w_1, \ldots, w_r \in E^1(M)$ and that $x \in M$. Suppose that $\alpha_1, \alpha_2, \ldots, \alpha_s$ are loops in M based at x.*

(a) *If $I \in B_r$ and $r < s$, then*

$$\langle I, (1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_s) \rangle = 0,$$

where 1 denotes η_x, the constant path at x.

(b) *If $I \in \overline{B}_r$ and $r < s$, then*

$$\langle I, [\alpha_1[\alpha_2[\ldots[\alpha_{s-1}, \alpha_s]\ldots]\rangle = 0.$$

PROOF. To prove (a), we need only consider the case where $I = \int w_1 \cdots w_r$, and $0 \leq r < s$. The case when $r = 0$ (i.e., $I = 1$) can be verified directly.

Suppose $1 \leq r < s$. Set

$$\omega = \begin{pmatrix} 0 & w_1 & 0 & \cdots & 0 \\ & 0 & w_2 & & \vdots \\ & & 0 & & 0 \\ & 0 & & & w_r \\ & & & & 0 \end{pmatrix}.$$

Then T is given by (2.7) and

$$\langle T, (1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_s)\rangle = T(1-\alpha_1)T(1-\alpha_2)\cdots T(1-\alpha_s).$$

But each $T(1-\alpha_j)$ is a nilpotent upper triangular $(r+1)\times(r+1)$ matrix. Since $s > r$, their product must vanish and $\langle T, (1-\alpha_1)\cdots(1-\alpha_s)\rangle = 0$. Examining the top right-hand entry of this matrix yields the result.

Assertion (b) follows similarly. The relevant fact is that each $T(\alpha_j)$ is a unipotent upper triangular $(r+1)\times(r+1)$ matrix. Any bracket arrangement of length $> r$ of such matrices is the identity. □

Next we describe how to pointwise multiply two iterated integrals. Recall that a permutation σ of $\{1, 2, \ldots, r+s\}$ is a *shuffle* of type (r, s) if

$$\sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(r)$$

and

$$\sigma^{-1}(r+1) < \sigma^{-1}(r+2) < \cdots < \sigma^{-1}(r+s).$$

The following property of iterated integrals is related to the Baker-Campbell-Hausdorff formula.

(2.11) LEMMA (REE [**27**]). *If $w_1, w_2, \ldots, w_{r+s} \in E^1(M)$ and $\alpha \in PM$, then*

$$\int_\alpha w_1 w_2 \cdots w_r \int_\alpha w_{r+1} w_{r+2} \cdots w_{r+s} = \sum_\sigma \int_\alpha w_{\sigma(1)} w_{\sigma(2)} \cdots w_{\sigma(r+s)},$$

where σ runs over the shuffles of type (r, s).

PROOF. This is an exercise using the fact that

$$\Delta^r \times \Delta^s = \bigcup_{\sigma \in \mathrm{Sh}(r,s)} \{(t_{\sigma(1)}, \ldots, t_{\sigma(r+s)}) : 0 \leq t_{\sigma(1)} \leq \cdots \leq t_{\sigma(r+s)} \leq 1\}.$$

Here we are identifying the standard n-simplex Δ^n with

$$\{(t_1, \ldots, t_n) : 0 \leq t_1 \leq \cdots \leq t_n \leq 1\}$$

and $\mathbf{R}^r \times \mathbf{R}^s$ with $\mathbf{R}^{r+s}$. □

For example,

$$\int_\alpha w_1 \int_\alpha w_2 = \int_\alpha w_1 w_2 + \int_\alpha w_2 w_1.$$

Two more properties of iterated integrals that will be used in the sequel follow. Their proofs are straightforward.

(2.12) PROPOSITION. *If* $w_1, \ldots, w_r \in E^1(M)$ *and* $\gamma \in PM$, *then*

$$\int_{\gamma^{-1}} w_1 \cdots w_r = (-1)^r \int_\gamma w_r \cdots w_1. \quad \square$$

(2.13) PROPOSITION. (a) *If* $w_1, w_2 \in E^1(M)$ *and* α_j, β_j $(j = 1, \ldots, g)$ *are loops based at* x, *then*

$$\left\langle \int w_1 w_2, \prod_{j=1}^g [\alpha_j, \beta_j] \right\rangle = \sum_{j=1}^g \begin{vmatrix} \int_{\alpha_j} w_1 & \int_{\beta_j} w_1 \\ \int_{\alpha_j} w_2 & \int_{\beta_j} w_2 \end{vmatrix}.$$

(b) *If* $w_1, \ldots, w_r \in E^1(M)$ *and* $\alpha_1, \ldots, \alpha_r$ *are loops based at* x, *then*

$$\left\langle \int w_1 \cdots w_r, \prod_{j=1}^r (\alpha_j - 1) \right\rangle = \prod_{j=1}^r \int_{\alpha_j} w_j. \quad \square$$

Of course the whole discussion is valid for $\mathbf{C}$-valued forms and connections on trivial complex vector bundles $\mathbf{C}^n \times M \to M$.

3. Homotopy functionals. In this section we show how to construct iterated integrals that are homotopy functionals. Suppose that ∇ is a connection on the trivial bundle $\mathbf{R}^n \times M \to M$. The associated transport function $T\colon PM \to \mathrm{GL}(n)$ is the homotopy functional if and only if ∇ is flat. Consequently, all of the entries of T are homotopy functionals $PM \to \mathbf{R}$ if and only if ∇ is flat. In general, these entries are infinite sums of iterated integrals as we have seen in (2.5). However, if the connection form ω of ∇ lies in $E^1(M) \otimes \mathfrak{n}$, where $\mathfrak{n}$ is a nilpotent sub-Lie algebra of $\mathrm{gl}(n)$, then the entries of T are iterated integrals.

The connection ∇ extends naturally to the p-forms that take values in $\mathbf{R}^n$ by

$$\nabla\varphi = d\varphi + (-1)^{p+1}\varphi \wedge \omega.$$

Now ∇ is flat if and only if $\nabla^2 \equiv 0$ and a short computation shows that $\nabla^2 \equiv 0$ if and only if ω is *completely integrable.* That is,

$$d\omega + \omega \wedge \omega = 0.$$

For example, suppose that

$$\omega = \begin{pmatrix} 0 & w_1 & w_{12} \\ 0 & 0 & w_2 \\ 0 & 0 & 0 \end{pmatrix}$$

in which case

$$d\omega + \omega \wedge \omega = \begin{pmatrix} 0 & dw_1 & w_1 \wedge w_2 + dw_{12} \\ 0 & 0 & dw_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus ω is completely integrable if and only if

$$dw_1 = dw_2 = 0 \quad \text{and} \quad w_1 \wedge w_2 + dw_{12} = 0.$$

The associated transport is

$$T = \begin{pmatrix} 1 & \int w_1 & \int w_1w_2 + w_{12} \\ 0 & 1 & \int w_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Consequently, if $dw_1 = dw_2 = 0$, then $\int w_1w_2 + w_{12}$ is a homotopy functional if and only if $w_1 \wedge w_2 + dw_{12} = 0$.

The next result generalizes this example and is an introduction to Chen's power series connections (cf. [**5**, **6**]).

(3.1) PROPOSITION. *Suppose that $w_1, \ldots, w_r$, $u \in E^1(M)$ and $a_{ij} \in \mathbf{R}$, $1 \le i, j \le r$. If each w_j is closed, then*

$$\sum a_{ij} \int w_iw_j + \int u$$

is a homotopy functional if and only if

$$du + \sum a_{ij} w_i \wedge w_j = 0.$$

PROOF. Denote the free associative algebra generated by $X_1, X_2, \ldots, X_r, Z$ by $R\langle X_1, \ldots, X_r, Z\rangle$. Let I be the ideal generated by $X_1, X_2, \ldots, X_r, Z$. Set

$$A = \frac{R\langle X_1, \ldots, X_r, Z\rangle}{(X_iX_j - a_{ij}Z) + I^3}.$$

Define an A-valued 1-form ω by

$$\omega = w_1X_1 + \cdots + w_rX_r + uZ.$$

Since the w_j are closed,

$$d\omega + \omega \wedge \omega = duZ + \sum w_i \wedge w_jX_iX_j = (du + \sum a_{ij}w_i \wedge w_j)Z,$$

and ω is completely integrable if and only if $du + \sum a_{ij} w_i \wedge w_j = 0$.

Note that A is finite-dimensional and that the algebra homomorphism $A \to \mathrm{gl}(A)$ given by right multiplication is injective. From (2.5) we have

$$\begin{aligned} T &= 1 + \int \omega + \int \omega\omega \\ &= 1 + \sum \int w_jX_j + \int uZ + \sum \int w_iw_jX_iX_j \\ &= 1 + \sum \int w_jX_j + \left(\sum a_{ij} \int w_iw_j + \int u\right) Z. \end{aligned}$$

Since each w_j is closed and since the X_j and Z are linearly independent in A, it follows that T is a homotopy functional if and only if

$$\sum a_{ij} \int w_iw_j + \int u$$

is. The result follows from the fact that T is a homotopy functional if and only if ω is completely integrable. □

Iterated integrals of algebraic 1-forms can also be defined. Suppose, for example, that M is a compact Riemann surface. Let w_1 and w_2 be abelian differentials of the second kind on M. Define

$$w_1 \cup w_2 = \sum a_i p_i, \qquad a_i \in \mathbf{C},\ p_i \in M$$

if, locally, $w_i = df$ and

$$\left(2\pi\sqrt{-1}\right)^{-1} \operatorname{Res}_{p_j} f w_2 = a_j.$$

If $w_1, \ldots, w_r$ are abelian differentials of the second kind on M such that

$$\sum a_{ij} w_i \cup w_j = \sum a_k p_k,$$

where $\sum a_k = 0$, then we can find an abelian differential of the third kind u such that

$$\left(2\pi\sqrt{-1}\right)^{-1} \operatorname{Res}_{p_j} u = -a_j$$

and u is holomorphic away from the p_j. The iterated integral

$$\sum \int a_{ij} w_i w_j + u$$

is easily seen to be a well-defined homotopy functional $PM \to \mathbf{C}$. This is the algebraic analogue of (3.1).

4. The π_1 de Rham theorem. In this section we prove Chen's de Rham theorem for the truncation of the real group ring $\mathbf{R}\pi_1(M,x)$ by a power of its augmentation ideal.

Suppose that G is a (discrete) group and that R is a ring. We shall denote the group algebra of G over R by RG and the kernel of the augmentation $RG \to R\colon g \to 1$ by J. The kth power of the augmentation ideal will be denoted by J^k.

Let M be a manifold and $x \in M$. Denote the set of elements of $B_s(M)$ that are homotopy functionals {loops in M based at x} $\to \mathbf{R}$ by $H^0(B_s(M),x)$.[4] Integration induces a linear map

$$H^0(B_s(M),x) \to \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}\pi_1(M,x),\mathbf{R}).$$

According to (2.10), each element of $H^0(B_s(M),x)$ vanishes on J^{s+1} so that we have a map

$$H^0(B_s(M),x) \to \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}\pi_1(M,x)/J^{s+1},\mathbf{R}).$$

(4.1) THEOREM (CHEN). *For each $s \geq 0$, the integration map*

$$H^0(B_s(M),x) \to \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}\pi_1(M,x)/J^{s+1},\mathbf{R})$$

is an isomorphism.

PROOF. We will prove the theorem under the extra assumption that $\pi_1(M,x)$ is finitely generated. Set $G = \pi_1(M,x)$ and $V = RG/J^{s+1}$. Since G is finitely generated, V is finite-dimensional. Define a representation $\rho\colon G \to \operatorname{Aut}(V)$ by $g \to \{A \to Ag\}$. (Here our linear maps act on the right of V.) The first step is to show that this representation is unipotent.

[4]This notation is reasonable as homotopy functionals {loops in M based in x} $\to \mathbf{R}$ are locally constant functions on {loops in M based at x}. If one defines the full complex of iterated integrals (cf. [**5**]) then this is indeed H^0 of a cochain complex.

(4.2) PROPOSITION. *The representation ρ satisfies $(\rho(g) - I)^{s+1} = 0$ for all g.*

PROOF. First note that the powers of J induce a filtration

$$(*) \qquad V = RG/J^{s+1} \geq J/J^{s+1} \geq \cdots \geq J^s/J^{s+1} \geq 0.$$

The result now follows because G acts trivially on the graded quotients J^t/J^{t+1} of V: If $u \in J^t$ and $g \in G$, then $u(g-1) \in J^{t+1}$ so that $ug \equiv u \bmod J^{t+1}$. □

Next form the flat bundle

$$\begin{array}{c} V \to E \\ \downarrow \\ M \end{array}$$

with monodromy representation ρ. That is, $E = (V \times \tilde{M})/G$, where $(v,m)\cdot g = (vg, g^{-1}m)$ and $\tilde{M} \to M$ is a universal covering. Since G stabilizes the flag $(*)$ above, this bundle is filtered by flat subbundles

$$E \supseteq E^1 \supseteq E^2 \supseteq \cdots \supseteq E^s \supseteq 0,$$

where $E^t \to M$ has fiber J^t/J^{s+1}.

(4.3) PROPOSITION. *There is a C^∞ trivialization*

$$\begin{array}{ccc} V \times M & \to & E \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

of $E \to M$ such that $E^t \to M$ corresponds to $J^t/J^{s+1} \times M \to M$ and such that the induced connection on $J^t/J^{t+1} \times M \to M$ is trivial.

PROOF. This is proved by induction on s. Consider the short exact sequence of flat bundles

$$0 \to E^s \to E \xrightarrow{p} E/E^s \to 0.$$

As a flat bundle E^s is trivial and, by induction, E/E^s has the desired type of trivialization. Now use a splitting of p to obtain the desired trivialization of E. □

Denote the Lie algebra of endomorphisms of V that preserve the flag $(*)$ by $\mathrm{End}_J(V)$. The connection form ω of the trivialization of $E \to M$ given by (4.3) satisfies $\omega \in E^1(M) \otimes \mathrm{End}_J(V)$. Since every element A of $\mathrm{End}_J(V)$ satisfies $A^{s+1} = 0$, it follows from (2.5) that its transport T satisfies $T \in B_s(M) \otimes \mathrm{End}(V)$. Since the connection is flat, each entry of T lies in $H^0(B_s(M), x)$. Furthermore, since, by construction, the monodromy representation is the composite

$$\pi_1(M,x) \to \mathbf{R}\pi_1(M,x)/J^{s+1} \hookrightarrow \mathrm{End}(V),$$

we have

$$T \in H^0(B_s(M)) \otimes \mathbf{R}\pi_1(M,x)/J^{s+1}$$

and integration $\gamma \to \langle T, \gamma \rangle$ induces the identity

$$\mathbf{R}\pi_1(M,x)/J^{s+1} \to \mathbf{R}\pi_1(M,x)/J^{s+1}.$$

It follows that the integration map

$$H^0(B_s(M)) \to \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}\pi_1(M,x)/J^{s+1}, \mathbf{R})$$

is surjective. But it is clearly injective. This completes the proof of Theorem (4.1). □

The theorem remains true if $\mathbf{R}$ is replaced by $\mathbf{C}$ throughout and $E^\cdot(M)$ is replaced by the complex $E^\cdot(M) \otimes \mathbf{C}$ of $\mathbf{C}$-valued forms on M. It is desirable to be able to replace $E^\cdot(M)$ by any sub d.g. algebra of $E^\cdot(M)$ that computes the de Rham cohomology of M.

Suppose that k is $\mathbf{R}$ or $\mathbf{C}$ and that $A^\cdot$ is a sub d.g. algebra of the complex $E^\cdot_k(M)$ of k-valued forms on M such that the inclusion $A^\cdot \to E^\cdot_k(M)$ is a quasi-isomorphism. Define $B_s(A^\cdot)$ to be the space of iterated integrals spanned by $\int w_1 \cdots w_r$ where each $w_j \in A^1$ and $0 \le r \le s$. For each $x \in M$, define $H^0(B_s(A^\cdot), x)$ to be the elements of $B_s(A^\cdot)$ whose restriction to {loops based at x} is a homotopy functional. We can now state a refined version of (4.1).

(4.4) THEOREM (CHEN). *Suppose that $A^\cdot$ is a sub d.g. algebra of $E^\cdot_k(M)$, where $k = \mathbf{R}$ or $\mathbf{C}$. If the inclusion $A^\cdot \to E^\cdot_k(M)$ is a quasi-isomorphism, then for each $x \in M$ and $s \ge 0$, the integration map*

$$H^0(B_s(A^\cdot), x) \to \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}\pi_1(M,x)/J^{s+1}, k)$$

is an isomorphism. □

(4.5) REMARKS. One can define iterated integrals of forms of arbitrary degree (see [**5**]). If $w_1, \ldots, w_r \in E^\cdot(M)$ are forms of positive degree, then $\int w_1 w_2 \cdots w_r$ is a differential form of degree $\sum(\deg w_j - 1)$ on the path space PM. In particular, if each w_j is a 1-form, then $\int w_1 \cdots w_r$ is a smooth function on PM. The space of iterated integrals $\int E^\cdot(M)$ forms a subcomplex of the de Rham complex of PM. When restricted to {loops at x}, the formula for the differential becomes

$$\begin{aligned} d\int w_1 \cdots w_r = &-\sum_{i=1}^{r} \int w_1 \cdots w_{i+1}\, dw_i w_{i+1} \cdots w_r \\ &-\sum_{i=1}^{r-1} \int w_1 \cdots w_{i-1}(w_i \wedge w_{i+1}) w_{i+2} \cdots w_r, \end{aligned}$$

when each w_j is a 1-form. For example, if each w_j is a closed 1-form, then

$$d\left(\sum a_{ij} \int w_i w_j + \int u\right) = -\sum a_{ij} \int w_i \wedge w_j - \int du.$$

This gives an alternative proof of (3.1).

The full complex of iterated integrals is a canonical quotient of the bar construction on the de Rham complex of $E^\cdot(M)$.

For Hodge theory it is convenient to state a variant of (4.4). Suppose that $A^\cdot$ and k are as in (4.4). As we have seen in §2, $B(A^\cdot) = k \oplus \overline{B}(A^\cdot)$. This decomposition restricts to give a decomposition

$$H^0(B(A^\cdot), x) = k \oplus H^0(\overline{B}(A^\cdot), x).$$

Denote the augmentation ideal of $\mathbf{Z}\pi_1(M, x)$ by $J(M, x)$. The kernel of the map

$$H^0(B_s(A^\cdot), x) \to \mathrm{Hom}(J(M, x)/J^{s+1}, k)$$

is k. The following result is a restatement of (4.4).

(4.6) THEOREM. *With the assumptions of* (4.4), *integration induces an isomrophism*

$$H^0(\overline{B}_s(A^\cdot), x) \to \mathrm{Hom}_{\mathbf{Z}}(J(M, x)/J^{s+1}, k).$$

5. The mixed Hodge structure on π_1. In this section we put a mixed Hodge structure (M.H.S.) on the truncated group ring $\mathbf{Z}\pi_1(V, x)/J^{s+1}$ (or, equivalently, its dual) of a smooth algebraic variety V with basepoint x. An equivalent result has been proved by Morgan [**20**, 9.2] using different techniques. Granted the π_1 de Rham theorem, (4.4), our construction of the M.H.S. on

$$\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}\pi_1(V, x)/J^{s+1}, \mathbf{C})$$

is direct and transparent. As we shall see in the proof of (5.1), an element φ of this group lies in F^p if it can be represented as a sum of iterated integrals $\int w_1 \cdots w_r$ each with at least p dz's. The weight of an iterated integral $\int w_1 \cdots w_r$ is its length r plus the number of logarithmic terms in the w_j.

(5.1) THEOREM. *If V is an algebraic variety over* $\mathbf{C}$ *and $x \in V$, then there is a M.H.S. on*

$$\mathbf{Z}\pi_1(V, x)/J^{s+1}$$

that is natural with respect to morphisms of pointed varieties. Moreover, if $s \geq t$, then the quotient map

$$\mathbf{Z}\pi_1(V, x)/J^{s+1} \to \mathbf{Z}\pi_1(V, x)/J^{t+1}$$

induces a morphism of M.H.S.'s.

We will sketch the proof when V is smooth. We begin by reviewing the construction of the M.H.S. on the cohomology of V. Suppose that V is a smooth quasi-projective variety and that X is a smooth projective completion of V such that $X - V$ is a divisor D in X with normal crossings. Denote the complex of C^∞ forms on V with logarithmic singularities along D by $E^\cdot(X \log D)$. (For a detailed description, see [**7**] or [**10**].) The inclusion $E^\cdot(X \log D) \to E^\cdot_{\mathbf{C}}(V)$ is a quasi-isomorphism. The Hodge and weight filtrations of the C^∞ log complex are defined by

$$F^p E^\cdot(X \log D) = \{\text{forms with } \geq p \ dz\text{'s}\},$$
$$W_l E^\cdot(X \log D) = \left\{\text{forms with } \leq l \ \frac{dz}{z}\text{'s}\right\},$$

and the Hodge and weight filtrations on the complex cohomology of V are defined by

$$W_l H^m(V;\mathbf{C}) = \operatorname{im}\{H^m(W_{l-m}E^{\cdot}(X \log D)) \to H^m(V)\},$$
$$F^p H^m(V;\mathbf{C}) = \operatorname{im}\{H^m(F^p E^{\cdot}(X \log D)) \to H^m(V)\}.$$

One can show that the weight filtration on $H^m(V;\mathbf{C})$ is defined over $\mathbf{Q}$.

Define the Hodge filtration of $B_s(E^{\cdot}(X \log D))$ by defining $F^p B_s(E^{\cdot}(X \log D))$ to be the subspace spanned by the iterated integrals $\int w_1 \cdots w_r$, where $w_j \in F^{p_j}$ and $p_1 + p_2 + \cdots + p_r \geq p$. The weight filtration of $B_s(E^{\cdot}(X \log D))$ is defined by letting $W_l B_s$ be the span of the $\int w_1 \cdots w_r$, where $w_j \in W_{l_j}$ and $l_1 + l_2 + \cdots + l_r + r \leq l$. That is, the weight of an iterated integral is its length plus the sum of the weights of the w_j's.

Note that if V is projective, then the weight filtration on $E^1(V)$ is $0 = W_{-1} \subseteq W_0 = E^1(V)$. It follows that if V is projective and smooth, then

$$W_l B^{\cdot}(E(V)) = \begin{cases} B_l(E^{\cdot}(V)) & \text{if } l \leq s, \\ B_s(E^{\cdot}(V)) & \text{if } l \geq s. \end{cases}$$

The Hodge and weight filtrations of $B_s(E^{\cdot}(X \log D))$ induce Hodge and weight filtrations on $H^0(B_s E^{\cdot}(X \log D), x)$ for each $x \in V$.

The theorem is proved by induction on s. The rational structure will be suppressed. We first need the following result.

(5.2) PROPOSITION. *There is a natural isomorphism*

$$\mathbf{C} \oplus H^1(V;\mathbf{C}) \to H^0(B_1(V), x),$$
$$(\lambda, \{w\}) \to \lambda + \int w.$$

PROOF. It follows from elementary properties of line integrals that $\int w$ is a homotopy functional if and only if w is closed. If $w = df$ and α is a loop based at x, then

$$\int_\alpha w = f \circ \alpha(1) - f \circ \alpha(0) = 0.$$

It follows that the map is well defined and an isomorphism. □

An immediate consequence of this result is that $H^0(B_1(V, x))$ has a M.H.S. For the inductive step we need the following result. According to (2.13)(b), we have a well-defined map

$$p\colon H^0(B_s(V), x) \to \bigotimes^s H^1(V;\mathbf{C})$$

that takes the iterated integral I to the function

$$\bigotimes^s H_1(V;\mathbf{C}) \to \mathbf{C},$$
$$[\alpha_1] \otimes \cdots \otimes [\alpha_s] \to \left\langle I, \prod_{j=1}^{s} (\alpha_j - 1) \right\rangle,$$

where each α_j is a loop based at x.

(5.3) PROPOSITION. *For all s, the sequence*

$$0 \to H^0(B_{s-1}(V), x) \to H^0(B_s(V), x) \xrightarrow{p} \bigotimes^s H^1(V, \mathbf{C})$$

is exact.

PROOF. We have to show that $H^0(B_{s-1}, x)$ is the kernel of p. Suppose that $I \in H^0(B_s, x)$. Write

$$I = \sum a_J \int w_{j_1} \cdots w_{j_s} + I',$$

where $a_J \in \mathbf{C}$ and $I' \in B_{s-1}$. Since I is a homotopy functional, it follows from (2.13)(b) that

$$\tilde{p}(I) = \sum a_J w_{j_1} \otimes \cdots \otimes w_{j_s}$$

is closed in $\bigotimes^s E^{\cdot}(V)$ and that $p(I)$ is its cohomology class. If $p(I) = 0$, then

$$\tilde{p}(I) = d\left(\sum b_K z_{k_1} \otimes \cdots \otimes z_{k_{i-1}} \otimes f_k \otimes z_{k_i} \otimes \cdots \otimes z_{k_{i-1}}\right),$$

where each $z_j \in E^1(V)$ and $f_k \in E^0(V)$. From (1.3) it follows that $I \in H^0(B_{s-1}, x)$. $\square$

One can show, with the aid of the Eilenberg-Moore spectral sequence, that the image of p is defined over $\mathbf{Q}$ and that it is a sub-M.H.S. of $\bigotimes^s H^1(V; \mathbf{C})$. For example, when V is projective or $W_1 H^1(V) = 0$, then the image of p is the kernel of the map

$$\sum_{i=1}^{s-1} c_i \colon \bigotimes^s H^1(V; \mathbf{C}) \to \sum_{i+j=s-2} \bigotimes^i H^1(V) \otimes H^2(V) \otimes \bigotimes^j H^1(V),$$

where

$$c_i(z_1 \otimes \cdots \otimes z_s) = z_1 \otimes \cdots \otimes z_i \otimes (z_i \wedge z_{i+1}) \otimes z_{i+2} \otimes \cdots \otimes z_s.$$

In this case it is clear that $\operatorname{im} p$ is defined over $\mathbf{Q}$ and is a sub-Hodge structure of $\bigotimes^s H^1(V)$.

In order to complete the proof, note that the Hodge and weight filtrations of $H^0(B_s(V), x)$ induce those of $H^0(B_{s-1}(V), x)$ and $\operatorname{im} p$. The result now follows from [**10**, 1.16]. $\square$

(5.4) REMARK. In the proof we have shown that if $H^1(V)$ is a pure Hodge structure, then the graded quotients of the weight filtration of $H^0(B_s(V), x)$ have natural polarizations. This follows because they are the sub-Hodge structures $\operatorname{im} p$ of $\bigotimes^s H^1(V)$.

As a M.H.S., the dual of the truncated group ring splits

$$H^0(B_s(V), x) = \mathbf{C} \oplus H^0(\overline{B}_s(V), x),$$

so that the interesting part of the M.H.S. is $H^0(\overline{B}_s(V), x)$. For this reason we record the following corollary of (4.6) and (5.1).

(5.5) COROLLARY. *For each $s \geq 0$ and $x \in V$, there is a natural M.H.S. on*

$$H^0(\overline{B}_s(V), x) \cong \operatorname{Hom}_{\mathbf{Z}}(J(V, x)/J^{s+1}, \mathbf{C}).$$

6. Extension data. From the point of view of Hodge theory, the interesting part of the group ring of the fundamental group of a smooth variety is its augmentation ideal J. The filtration $J \geq J^2 \geq J^3 \geq \cdots$ of J by its powers is closely related to the filtration of the fundamental group by its lower central series (see [**23**]). For example, the function

$$\begin{aligned} \pi_1(V,x) &\to J/J^2, \\ g &\to \overline{g-1} \end{aligned}$$

induces a group isomorphism $H_1(V;\mathbf{Z}) \to J/J^2$. Thus the first interesting quotient of J should be J/J^3. When $H_1(V;\mathbf{Z})$ is torsionfree, there is an exact sequence (over $\mathbf{Z}$)

$$0 \to H^1(V) \to \mathrm{Hom}(J/J^3, \mathbf{Z}) \to K \to 0,$$

where K is the kernel of the cup product $H^1(V) \otimes H^1(V) \to H^2(V)$. When $H^1(V)$ has a pure Hodge structure, the M.H.S. on the dual of J/J^3 is a separated extension of Hodge structures. In this section we review some extension theory of Hodge structures and give a formula for the extension data for J/J^3. Full details of the extension theory can be found in [**3**].

A *separated extension of Hodge structures* is an exact sequence

$$0 \to A \to E \to B \to 0$$

of M.H.S.'s, where A is a pure Hodge structure of weight m, B is a pure Hodge structure of weight n, and $n > m$. Two extensions

$$0 \to A \to E_j \to B \to 0, \qquad j = 1,2,$$

are *congruent* if there is an isomorphism of M.H.S.'s $\Phi\colon E_1 \to E_2$ such that

$$\begin{array}{ccccccccc} 0 \to & A & \to & E_1 & \to & B & \to 0 \\ & \mathrm{id}\downarrow & & \downarrow\Phi & & \downarrow\mathrm{id} & \\ 0 \to & A & \to & E_2 & \to & B & \to 0 \end{array}$$

commutes. Note that congruence is a finer equivalence relation than isomorphism, where Φ would be allowed to induce any automorphisms of the Hodge structures A and B.

The set of congruence classes of extensions of B by A forms an abelian group that we shall denote by $\mathrm{Ext}(B,A)$. Let $F^0\,\mathrm{Hom}_{\mathbf{C}}(B,A)$ be the set of Hodge filtration preserving $\mathbf{C}$-linear maps $B \to A$. There is an abelian group isomorphism

$$\psi\colon \mathrm{Ext}(B,A) \to \frac{\mathrm{Hom}_{\mathbf{C}}(B,A)}{F^0\,\mathrm{Hom}_{\mathbf{C}}(B,A) + \mathrm{Hom}_{\mathbf{Z}}(B,A)}$$

that is given as follows. If

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"'] & E \arrow[l, bend right, "r_{\mathbf{Z}}"'] \arrow[r, "p"'] & B \arrow[l, bend right, "s_F"'] \arrow[r] & 0 \end{tikzcd}$$

is an extension, choose a Hodge filtration preserving section $s_F: B \to E$ of p and a retraction $r_{\mathbf{Z}}: E \to A$ of i that is defined over $\mathbf{Z}$. Composing these gives an element $\psi(E) = r_{\mathbf{Z}} \circ s_F$ of $\mathrm{Hom}_{\mathbf{C}}(B, A)$. One can check that $\psi(E)$ is well defined modulo $F^0 \operatorname{Hom}(B, A) + \operatorname{Hom}_{\mathbf{Z}}(B, A)$. It is not hard to construct an inverse of ψ. For details, see [**3**].

When B is of weight $2p$, then the extension homomorphism $r_{\mathbf{Z}} \circ s_F$ also induces a group homomorphism

$$\mu \colon B_{\mathbf{Z}}^{p,p} \to A_{\mathbf{C}}/F^p A + A_{\mathbf{Z}}$$

from the integral points of type (p,p), $B_{\mathbf{Z}} \cap B^{p,p}$, of B into a complex torus. This is called the *motive* of the extension (cf. [**3**]).

Returning to the case of interest, we first express the M.H.S. on $(J/J^3)^*$ as an extension.

(6.1) LEMMA. *Suppose that (X, x) is a path connected, pointed topological space. If $H_1(X; \mathbf{Z})$ is torsionfree, then there is an exact sequence*

$$0 \to H^1_{\mathbf{Z}}(X) \xrightarrow{i} \operatorname{Hom}_{\mathbf{Z}}(J(X,x)/J^3, \mathbf{Z}) \xrightarrow{p} H^1_{\mathbf{Z}}(X) \otimes H^1_{\mathbf{Z}}(X) \to H^2_{\mathbf{Z}}(X).$$

Here $i(z)(g-1) = \langle z, g \rangle$, where $g \in \pi_1(X, x)$ and $z \in H^1(X)$. If $\varphi \in (J/J^3)^*$ and α, β are loops based at x, then

$$p(\varphi)([\alpha] \otimes [\beta]) = \langle \varphi, (\{\alpha\} - 1)(\{\beta\} - 1) \rangle.$$

The reader may enjoy trying to prove this when X is a smooth curve over $\mathbf{C}$. The general result can be proved using the cobar construction [**28**].

Now suppose that V is a smooth variety $/\mathbf{C}$ and that $x \in V$. For the rest of this section we suppose that $H^1(V)$ has a pure Hodge structure of weight l, necessarily 1 or 2. Denote the kernel of the cup product $H^1(V) \otimes H^1(V) \to H^2(V)$ by K. Since the cup product is a morphism of Hodge structures, K is a polarized Hodge structure of weight $2l$. Thus we have a separated extension of Hodge structures

$$0 \to H^1(V) \xrightarrow{i} (J(V,x)/J^3)^* \xrightarrow{p} K \to 0.$$

If $l = 2$, we have a motive

$$\mu \colon K_{\mathbf{Z}} \to H^1_{\mathbf{C}}(V)/H^1_{\mathbf{Z}}(V).$$

To define an integral retraction

$$r_{\mathbf{Z}} \colon (J(V,x)/J^3)^* \to H^1(V),$$

choose elements $\alpha_1, \ldots, \alpha_n$ of $\pi_1(V, x)$ whose homology classes $[\alpha_1], \ldots, [\alpha_n]$ form a basis of $H_1(V; \mathbf{Z})$. Define

$$r_{\mathbf{Z}}(\varphi)[\alpha_j] = \varphi(\alpha_j - 1).$$

Let X be a smooth completion of V such that $D = X - V$ is a divisor with normal crossings. One can show that the natural map

$$(*) \qquad H^{\cdot}(F^p E^{\cdot}(X \log D)) \to H^{\cdot}(E^{\cdot}(X \log D))$$

is injective. Now, if

$$\sum a_{ij} z_i \otimes z_j \in F^p K_{\mathbf{C}},$$

then we can choose closed 1-forms $w_j \in E^1(X \log D)$ such that $[w_j] = z_j$ and

$$\sum a_{ij} w_i \otimes w_j \in F^p \bigotimes^2 E^1(X \log D).$$

Since $\sum a_{ij} z_i \wedge z_j = 0$, it follows from $(*)$ that there exists $u \in F^p E^1(X \log D)$ such that $du + \sum a_{ij} w_i \wedge w_j = 0$. According to (3.1), the iterated integral

$$\sum a_{ij} \int w_i w_j + \int u$$

is a homotopy functional. Define

$$s_F(\sum a_{ij} z_i \otimes z_j) = \sum a_{ij} \int w_i w_j + \int u \in F^p H^0(\overline{B}_2(V)).$$

From (2.13)(b) it follows that $p \circ s_F$ is the identity on K.

Combining these we have the following fact.

(6.2) PROPOSITION. *With the notation above, the extension homomorphism ψ associated with the M.H.S. on $(J(V,x)/J^3)^*$ is*

$$\psi\left(\sum a_{ij} z_i \otimes z_j\right)[\alpha_k] = \int_{\alpha_k} \left(\sum a_{ij} w_i w_j + u\right). \quad \square$$

We conclude this section with an example.

(6.3) EXAMPLE. Let $V = \mathbf{P}^1 - \{a_1, a_2, a_3\}$. We will compute the motive

$$\mu(t) \colon K_{\mathbf{Z}} \to H^1_{\mathbf{C}}(V)/H^1_{\mathbf{Z}}(V)$$

associated with the basepoint $t \in V$. Note that $\pi_1(V, t)$ is a free group on two generators. Set $D = \{a_1, a_2, a_3\}$. The Gysin homomorphism gives a canonical isomorphism $H^1(V) \cong H^0(D)$. The polarization of $H^0(D)$ given by $\langle a_i^*, a_j^* \rangle = \delta_{ij}$ induces the polarization of $H^1_{\mathbf{Z}}$ which distinguishes, up to a permutation of $\{a_1, a_2, a_3\}$, the integral basis

$$w_j = \frac{1}{2\pi\sqrt{-1}} d \log\left(\frac{x - a_j}{x - a_3}\right), \qquad j = 1, 2,$$

of $H^1(V)$.

To compute the extension data, choose loops γ_1, γ_2 based at t such that $\int_{\gamma_i} w_j = \delta_{ij}$. We may suppose that γ_i is nullhomotopic in $\mathbf{P}^1 - \{a_j, a_3\}$ when $j \neq i, 3$. Since there are no holomorphic 2-forms on V, $K = H^1 \otimes H^1$ and we can define s_F by

$$s_F\left(\sum a_{ij} w_i \otimes w_j\right) = \sum a_{ij} \int w_i w_j.$$

The ordered basis w_1, w_2 of $H^1_{\mathbf{Z}}(V)$ gives a canonical identification of $H^1_{\mathbf{Z}}(V)$ with $\mathbf{Z}^2$. The exponential map then gives a canonical identification

$$\begin{aligned} &H^1_{\mathbf{C}}(V)/H^1_{\mathbf{Z}}(V) \to \mathbf{C}^* \times \mathbf{C}^*, \\ &a_1 w_1 + a_2 w_2 \to (\exp 2\pi\sqrt{-1} a_1, \exp 2\pi\sqrt{-1} a_2). \end{aligned}$$

Since, by (2.11),

$$\int_{\gamma_k} w_j w_j = \frac{1}{2}\left(\int_{\gamma_k} w_j\right)^2 = \frac{1}{2}\delta_{jk}$$

we have

$$\mu(w_j \otimes w_j)[\gamma_k] = \exp(\pi\sqrt{-1}\delta_{jk}) = \begin{cases} -1, & j = k, \\ 1, & j \neq k. \end{cases}$$

This is not very interesting as it does not depend upon the basepoint. Again, by (2.11),

$$\int_{\gamma_k} w_i w_j + \int_{\gamma_k} w_j w_i = \int_{\gamma_k} w_i \int_{\gamma_k} w_j \in \mathbf{Z},$$

so that

$$\mu(w_i \otimes w_j)[\gamma_k] = (\mu(w_j \otimes w_i)[\gamma_k])^{-1}.$$

Thus we need only compute $\mu(w_1 \otimes w_2)[\gamma_2]$ and $\mu(w_2 \otimes w_1)[\gamma_1]$. Now

$$\begin{aligned} \mu(w_1 \otimes w_2)[\gamma_2] &= \exp\left\{2\pi\sqrt{-1}\int_{\gamma_2} w_1 w_2\right\} \\ &= \exp\left\{\frac{1}{2\pi\sqrt{-1}}\int_{\gamma_2} \log\left[\left(\frac{x-a_1}{x-a_3}\right)\Big/\left(\frac{t-a_1}{t-a_3}\right)\right] d\log\left(\frac{x-a_2}{x-a_3}\right)\right\} \\ &= \exp\left\{\log\left[\frac{(a_1-a_2)/(a_3-a_2)}{(a_1-t)/(a_3-t)}\right]\right\} = [a_1, a_3, a_2, t], \end{aligned}$$

the cross ratio λ of the four points a_1, a_3, a_2, t. Consequently, the congruence class of the M.H.S. on $J(V,t)/J^3$ determines the ordered 4-tuple (a_1, a_2, a_3, t).

Since the choice of w_1 and w_2 is only defined up to a permutation of $\{a_1, a_2, a_3\}$, all six possible values of the cross ratio

$$\lambda, \quad \lambda^{-1}, \quad 1-\lambda, \quad (1-\lambda)^{-1}, \quad \lambda(\lambda-1)^{-1}, \quad \lambda^{-1}(\lambda-1)$$

may occur. It follows that the isomorphism class of the polarized M.H.S. on $J(V,t)/J^3$ determines (V,t) up to biholomorphism. □

More generally, we have the following result:

(6.4) THEOREM. *If $V = P^1 - \{a_1, \ldots, a_n\}$, then the polarized M.H.S. on $J(V,t)/J^3$ determines (V,t) up to biholomorphism.*

This is the first hint that the M.H.S. on the fundamental group might be interesting and our first Torelli theorem.

(6.5) REMARK. Suppose that X is a compact Riemann surface of genus $g \geq 2$. Choose a basis of abelian differentials $w_1, \ldots, w_g$ on X. Gunning [**11**] has defined certain quadratic periods $Q_{ij}(\gamma)$, where γ is a loop in X based at x. In terms of our notation,

$$Q_{ij}(\gamma) = \int_\gamma w_i w_j.$$

Various properties of quadratic periods such as

$$Q_{ij}(\gamma) + Q_{ji}(\gamma) = \int_\gamma w_i \int_\gamma w_j$$

follow from properties of iterated integrals (2.11), (2.12), (2.13). From our viewpoint, the quadratic periods are giving the Plucker coordinates $F^2(J(X,x)/J^3)^*$ in $(J(X,x)/J^3)^*$ as

$$Q_{ij}(\gamma) = s_F(w_i \otimes w_j)[\gamma].$$

(6.6) REMARK. It is shown in **[14]** that the M.H.S. on $\mathbf{Z}\pi_1(V,x)/J^{s+1}$ depends holomorphically on the pair (V,x). When V is fixed, the holomorphic dependence of the Hodge filtration on the basepoint x follows by differentiating the change of basepoint formula

$$\int_{\gamma\alpha\gamma^{-1}} w_1 \cdots w_r = \sum_{\substack{i\geq 0\\ j\geq 0}} \int_\gamma w_1 \cdots w_i \int_\alpha w_{i+1} \cdots w_j \int_{\gamma^{-1}} w_{j+1} \cdots w_r$$

along γ.

7. Torelli theorems. In §6 we saw that the M.H.S. on $J(V,x)/J^3$ varies holomorphically with (V,x) and that its isomorphism class determines (V,x) up to isomorphism when $V = \mathbf{P}^1 - \{a_1,\dots,a_n\}$. In this section we give general conditions under which the period mapping

$$V \to \operatorname{Ext}(K, H^1(V)),$$
$$x \to \text{M.H.S. on } J(V,x)/J^3$$

is injective in the case when $H^1(V)$ has a pure Hodge structure. In certain cases this implies (almost) global Torelli theorems for (V,x) such as when V is a smooth projective curve.

Suppose that V is a smooth variety, and that $H^1(V)$ has a pure Hodge structure of weight l. To simplify the discussion, we also suppose that $H_1(V;\mathbf{Z})$ is torsionfree. Being the dual of $H^1(V)$, $H_1(V)$ has a Hodge structure of weight $-l$. Define the Albanese of V by

$$\operatorname{Alb}(V) = \frac{H_1(V;\mathbf{C})}{F^0H_1(V) + H_1(V;\mathbf{Z})}.$$

If X is a smooth projective completion of V such that $D = X - V$ is a divisor with normal crossings, then integration induces an isomorphism

$$\operatorname{Alb}(V) \to \frac{\operatorname{Hom}(\Omega^1(X \log D), \mathbf{C})}{H_1(V;\mathbf{Z})}$$

of complex tori. Fix a basepoint $x \in V$. We have a holomorphic map

$$\theta_x \colon V \to \operatorname{Alb}(V),$$
$$y \to \int_x^y \cdot$$

Define a map

$$\tilde{\Phi} \colon H_1(V;\mathbf{Z}) \to \operatorname{Hom}_{\mathbf{Z}}(K, H^1(V))$$

by

$$\tilde{\Phi}(A)\left(\sum a_{ij} z_i \otimes z_j\right)(B) = -\sum a_{ij} \begin{vmatrix} \langle z_i, A\rangle & \langle z_j, A\rangle \\ \langle z_i, B\rangle & \langle z_j, B\rangle \end{vmatrix},$$

where $A, B \in H_1(V;\mathbf{Z})$, $z_i \in H^1(V;\mathbf{Z})$, and $a_{ij} \in \mathbf{Z}$. One can easily check that $\tilde{\Phi}$ is a morphism of Hodge structures. Consequently, $\tilde{\Phi}$ induces a group homomorphism

$$\Phi\colon \mathrm{Alb}(V) \to \mathrm{Ext}(K, H^1(V)).$$

Denote by $m\colon V \to \mathrm{Ext}(K, H^1(V))$ the map that takes the point y of V to the congruence class of the M.H.S. on $J(V,y)/J^3$. Denote by ν_x by the map

$$\nu_x\colon V \to \mathrm{Ext}(K, H^1(V)),$$
$$y \to m(y) - m(x).$$

(7.1) PROPOSITION. *The diagram*

$$\begin{tikzcd} \mathrm{Alb}(V) \arrow[rr, "\Phi"] & & \mathrm{Ext}(K, H^1(V)) \\ & V \arrow[ul, "\theta_x"] \arrow[ur, "\nu_x"'] & \end{tikzcd}$$

commutes.[5]

PROOF. Choose a path γ in V from x to y. If $\alpha_1, \ldots, \alpha_n$ are elements of $\pi_1(V,x)$ whose homology classes form a basis of $H_1(V;\mathbf{Z})$, then the paths $\gamma^{-1}\alpha_j\gamma$ $(j = 1, \ldots, n)$ are elements of $\pi_1(V,y)$ with the same property. The result now follows from (6.2) and the following easily verified change of basepoint formula.

(7.2) PROPOSITION. *Suppose that $w_1, w_2 \in E^1(V)$. If α is a loop in V based at x and γ is a path in V from x to y, then*

$$\int_{\gamma^{-1}\alpha\gamma} w_1 w_2 = \int_\alpha w_1 w_2 - \begin{vmatrix} \int_\gamma w_1 & \int_\gamma w_2 \\ \int_\alpha w_1 & \int_\alpha w_2 \end{vmatrix}. \quad \square$$

To determine when the congruence class of the M.H.S. on $J(V,y)/J^3$ determines y is to determine when ν_x is injective. Since ν_x factors as in (7.1), the injectivity of θ_x and Φ will guarantee the injectivity of ν_x. We first study Φ.

For each $u \in H^1_{\mathbf{Q}}(V)$ set

$$u \wedge H^1(V) = \{u \wedge v \in \Lambda^2 H^1_{\mathbf{Q}}(V) \colon v \in H^1_{\mathbf{Z}}(V)\}.$$

(7.3) LEMMA. *The map Φ is finite to one if and only if for all $u \in H^1_{\mathbf{Q}}(V)$ the cup product*

$$u \wedge H^1_{\mathbf{Q}}(V) \to H^2_{\mathbf{Q}}(V)$$

is not injective. Moreover, if in addition $H_1(V;\mathbf{Z})$ is torsionfree, then Φ is injective.[6]

Note that since Φ is a group homomorphism, it is a submersion onto its image.

[5] Pulte [**25**] first observed this for smooth projective curves. This proposition is a very special case of the classification of variations of M.H.S. with unipotent monodromy given in [**14**].

[6] See note added in proof, page 280.

PROOF. Since $\tilde{\Phi}$ is a morphism of Hodge structures, and because both $H_1(V)$ and $\mathrm{Hom}(K, H^1(V))$ satisfy $F^0 \cap \overline{F}^0 = 0$, it follows that Φ is a covering if and only if

$$\Phi_{\mathbf{Q}}\colon H_1(V;\mathbf{Q}) \to \mathrm{Hom}_{\mathbf{Q}}(K, H^1(V))$$

is injective.

We can write K as a direct sum of skew symmetric tensors E plus symmetric tensors S. Now $\tilde{\Phi}$ is injective

$\Leftrightarrow$ $H_1(V;\mathbf{Q}) \to \mathrm{Hom}(E, H^1(V))$ is injective

$\Leftrightarrow$ The bilinear form

$$\det\colon H_1(V) \otimes H_1(V) \to E^*,$$

$$A \otimes B \to \left\{\sum a_{ij} z_i \wedge z_j \to \sum a_{ij} z_i(A) z_j(B)\right\}$$

is nondegenerate

$\Leftrightarrow$ For each nonzero element u of $H_1(V)$, the map $\det : u \wedge H_1(V) \to E^*$ is nonzero

$\Leftrightarrow$ For all nonzero z in $H^1(V)$, $z \wedge H^1(V) \cap E \neq 0$

$\Leftrightarrow$ For all $z \in H^1(V)$, the cup product $z \wedge H^1_{\mathbf{Q}}(V) \to H^2_{\mathbf{Q}}(V)$ is not injective.

The second assertion follows from (6.1). $\square$

The injectivity of $\theta_x\colon V \to \mathrm{Alb}(V)$ is sometimes well understood. For example, if V is a smooth projective curve of genus ≥ 2, then θ_x is injective and the conditions of (7.3) are satisfied so that Φ is injective. Thus we have proved

(7.4) LEMMA. *If V is a smooth projective curve of genus ≥ 2, then the congruence class of the mixed Hodge structure on $J(V,x)/J^3$ determines x. That is, $m(x) = m(y)$ if and only $x = y$.* $\square$

A nicer result would assert that the isomorphism class of the M.H.S. on $J(V,x)/J^3$ determines (V,x) up to isomorphism. This may well be true. However the following result is the best to date.

(7.5) THEOREM (HAIN, PULTE). *Suppose that (V,x) and (W,y) are two pointed smooth projective curves. If there is a ring isomorphism*

$$\varphi\colon \mathbf{Z}\pi_1(V,x)/J^3 \to \mathbf{Z}\pi_1(W,y)/J^3$$

that induces an isomorphism of M.H.S., then there is an isomorphism $f\colon V \to W$ such that, with the possible exception of at most two points x of V, $f(x) = y$. (If V is hyperelliptic, then no such exceptional points exist.)[7]

PROOF. Since φ induces an isomorphism of M.H.S.'s it will induce an isomorphism of Hodge structures on $W_{-1}/W_{-2} = J/J^2 \cong H_1$. Consequently, it induces an isomorphism of Hodge structures $\varphi^*\colon H^1(W) \to H^1(V)$. According to (6.1), the sequence

$$0 \to H^1(V) \to (J/J^3)^* \xrightarrow{p} H^1(V) \otimes H^1(V) \xrightarrow{\text{cup}} \mathbf{Z} \to 0 \quad \text{is exact.}$$

[7]Pulte [25] has shown that if the pointed Torelli theorem fails for X, a curve of genus g, then $(g-1)(p+q)$ is the canonical divisor of X, where $\{p,q\}$ is the unique pair of points such that $m(p) + m(q) = 0$.

Since p is dual to the multiplication

$$J/J^2 \otimes J/J^2 \to J/J^3,$$

the ring structure of $\mathbf{Z}\pi_1/J^3$ determines the polarization on H^1. Thus the fact that φ is a ring homomorphism implies that $\varphi^*\colon H^1(W) \to H^1(V)$ is an isomorphism of polarized Hodge structures.

By the classical Torelli theorem, there is an isomorphism $f\colon V \to W$ such that $f^*\colon H^1(W) \to H^1(V)$ is $\pm\varphi^*$. Since φ is a ring homomorphism, the induced map on $K = (J^2/J^3)^*$ will be $f^* \otimes f^* = \varphi^* \otimes \varphi^*$. Note that if V is hyperelliptic, we may assume that $f^* = \varphi^*$, for we may compose f with the hyperelliptic involution if necessary.

We may now assume that $V = W$ and that the diagram

$$\begin{array}{ccccc} 0 \to H^1(V) & \to & (J(V,y)/J^3)^* & \to & K \to 0 \\ \pm\mathrm{id} \downarrow & & \varphi^* \downarrow & & \downarrow \mathrm{id} \\ 0 \to H^1(V) & \to & (J(V,x)/J^3)^* & \to & K \to 0 \end{array}$$

commutes. That is, the congruence classes $m(x)$ and $m(y)$ of the M.H.S.'s on $(J(V,x)/J^3)^*$ and $(J(V,y)/J^3)^*$ satisfy $m(x) = \pm m(y)$. If $m(x) = m(y)$, then $x = y$ by (7.4). Now, if V is not hyperelliptic, then there is at most one pair of points $\{x, y\}$ of V such that $m(x) = -m(y)$. For if $m(x)+m(y) = m(p)+m(q) = 0$, then it follows from (7.1) and Abel's theorem that $x + y = p + q$ in $\mathrm{Pic}(V)$. Since V is not hyperelliptic, this implies that $\{x, y\} = \{p, q\}$. $\square$

Jablow [18] has proved a pointed Torelli theorem for a general pointed curve of genus 3 with a framing of H_1.

Even in the case when

$$\Phi\colon \mathrm{Alb}(V) \to \mathrm{Ext}(K, H^1)$$

is not an immersion, the pointed Torelli theorem may be true. For example, let V be the set of ordered 3-tuples of distinct points in $\mathbf{C}$. That is,

$$V = \mathbf{C} \times \mathbf{C} \times \mathbf{C} - \{(x,y,z)\colon (x-y)(y-z)(z-x) = 0\}.$$

When

$$u = \frac{d(x-y)}{x-y} + \frac{d(y-z)}{y-z} + \frac{d(z-x)}{z-x},$$

the cup product $u \wedge H^1(V) \to H^2(V)$ is an isomorphism. By (7.3), Φ is not a covering map. (In fact, $\Phi^{-1}(0) \cong \mathbf{C}^*$.) Let $x = (a_1, a_2, a_3)$. It is an interesting exercise to check that the extension data of $J(V,x)/J^3$ determines the cross ratio $[a_1, a_2, a_3, \infty]$. Consequently, the M.H.S. on $J(V,x)/J^3$ determines (V, x) up to isomorphism. The fibers of ν_x are the orbits of the action of the affine group

$$\begin{pmatrix} \mathbf{C}^* & \mathbf{C} \\ 0 & 1 \end{pmatrix}$$

on V:

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : \mathbf{x} \to a\mathbf{x} + (b,b,b), \qquad \mathbf{x} \in V.$$

8. Harmonic volume. Let C be a smooth projective curve over $\mathbf{C}$ of genus $g \geq 2$. For each $x \in C$ we have the Abel-Jacobi map

$$\begin{aligned} \theta_x : C &\to \mathrm{Jac}(C), \\ y &\to y - x. \end{aligned}$$

The image of θ_x is an algebraic 1-cycle C_x in $\mathrm{Jac}(C)$. Set

$$C_x^- = \{-p \in \mathrm{Jac}(C) : p \in C_x\}.$$

Since the involution $p \to -p$ of $\mathrm{Jac}(C)$ acts trivially on $H^2(\mathrm{Jac}(C))$, the algebraic 1-cycles $C_x - C_y$ and $C_x - C_y^-$ are homologous to zero.

Denote the group of algebraic k-cycles in the variety V, modulo rational equivalence, by $A_k(V)$ and those homologous to zero by $A_k^0(V)$. Abbreviating $\mathrm{Jac}(C)$ by Jac, we have

$$[C_x - C_y], [C_x - C_y^-] \in A_1^0(\mathrm{Jac}).$$

The intermediate Jacobian of Jac is defined by

$$J_2(\mathrm{Jac}) = \frac{\mathrm{Hom}(F^2H^3(\mathrm{Jac}), \mathbf{C})}{H_3(\mathrm{Jac}; \mathbf{Z})}.$$

There is an Abel-Jacobi map

$$\begin{aligned} \psi : A_1^0(\mathrm{Jac}) &\to J_2(\mathrm{Jac}), \\ [Z] &\to \textstyle\int_\Gamma, \end{aligned}$$

where $\partial\Gamma = Z$. The factorization of ν_x given in (7.1) has geometric meaning. First we need the following fact from linear algebra due to Pulte [**25**].

(8.1) PROPOSITION. *The natural map*

$$\begin{aligned} \textstyle\bigotimes^3 H^1_{\mathbf{Z}}(C) &\to \textstyle\bigwedge^3 H^1_{\mathbf{Z}}(C), \\ a \otimes b \otimes c &\to a \wedge b \wedge c \end{aligned}$$

induces a surjection

$$K_{\mathbf{Z}} \otimes H^1_{\mathbf{Z}}(C) \to \textstyle\bigwedge^3 H^1_{\mathbf{Z}}(C). \quad \square$$

(8.2) COROLLARY. *There is a natural injective group homomorphism*

$$J_2(\mathrm{Jac}) \to \mathrm{Ext}(K, H^1(C)).$$

PROOF. First note that Poincaré Duality

$$\mathrm{P.D.} : H_1(C) \to H^1(C)$$

is a morphism of Hodge structures of type $(1,1)$. Consequently, the isomorphism

$$\mathrm{Hom}(K, H^1(C)) \to \mathrm{Hom}(K \otimes H^1(C), \mathbf{C})$$

is of type $(-1,-1)$ and

$$\mathrm{Ext}(K, H^1(C)) \cong \frac{\mathrm{Hom}_{\mathbf{C}}(K \otimes H^1(C), \mathbf{C})}{\mathrm{Hom}(K \otimes H^1(C), \mathbf{Z}) + F^{-1}\,\mathrm{Hom}(K \otimes H^1, \mathbf{C})}.$$

Since $F^{-1}\,\mathrm{Hom}(K \otimes H^1, \mathbf{C})$ is the annihilator of $F^2(K \otimes H^1)$, it follows that

$$\mathrm{Ext}(K, H^1(C)) \cong \frac{\mathrm{Hom}(F^2(K \otimes H^1), \mathbf{C})}{\mathrm{Hom}_{\mathbf{Z}}(K \otimes H^1_{\mathbf{Z}}, \mathbf{Z})}.$$

The result now follows from (8.1). □

The following result gives a geometric interpretation of the factorization of ν_x given in (7.1). It is due to Pulte [**25**] who exploits techniques of B. Harris [**15**, **16**].

(8.3) THEOREM (HARRIS-PULTE). *The following diagram commutes*

$$\begin{tikzcd}
& J_2(\mathrm{Jac}) \arrow[dr, hook] & \\
C \arrow[ur, "\Psi_x"] \arrow[rr, "\nu_x"] \arrow[dr, "\theta_x"'] & & \mathrm{Ext}(K, H^1(C)) \\
& \mathrm{Jac}(C) \arrow[uu, "A"'] \arrow[ur, "\Phi"'] &
\end{tikzcd}$$

where A is the Abel-Jacobi map associated with the family

$$\begin{array}{ccccccc} C \times C & \to & \mathrm{Jac}(C) & \qquad & (y, z) & \to & y - z \\ \downarrow & & & & \downarrow & & \\ C & & & & z & & \end{array}$$

Also $\Psi[C_x - C_y^-] = m(x) + m(y)$. □

So far we have not mentioned B. Harris's harmonic volume explicitly. The technical ingredients of its construction are buried in the proof of (8.3). The harmonic volume of C, as defined by Harris, may be recovered as follows. Denote the primitive cohomology of $\mathrm{Jac}(C)$ by $PH^{\cdot}(\mathrm{Jac})$. There is a primitive intermediate Jacobian

$$P_2(\mathrm{Jac}) = \frac{\mathrm{Hom}(F^2 PH^3(\mathrm{Jac}), \mathbf{C})}{H_3(\mathrm{Jac}; \mathbf{Z})}$$

and a restriction map

$$r \colon J_2(\mathrm{Jac}) \to P_2(\mathrm{Jac}).$$

From (8.3) we know that $2m(x) \in J_2(\mathrm{Jac})$. Thus we can restrict $2m(x)$ to the primitives. One can show, either by using properties of iterated integrals or the fact that $2m(x) = \Psi[C_x - C_x^-]$, that $r(2m(x))$ is independent of x. Harris's definition of harmonic volume is equivalent to the following.

(8.4) DEFINITION. The harmonic volume $V(C)$ of C is the point $r(2m(x))$ of $P_2(\mathrm{Jac})$.

Harris's main result, which has been incorporated into (8.3), is that $r \circ \Psi[C_x - C_x^-]$ can be expressed as an iterated integral. Using this he showed that when C is the Fermat quartic, $C - C^-$ is not algebraically equivalent to zero. Ceresa [**4**] had previously proved that for the generic curve C of genus $g \geq 3$, $C - C^-$ is not algebraically equivalent to zero. The Fermat quartic was the first specific curve for which this was known. This played a role in Bloch's work [**2**].

It is interesting to note that Clemens has asked the following question.

(8.5) QUESTION. Is it true that a curve C of genus 3 is algebraically equivalent to its negative C^- in $\mathrm{Jac}(C)$ if and only if it is hyperelliptic? More generally, is it true that a curve C of genus $g = 2k - 1$ has the property that $C^{(g-1)}$ is algebraically equivalent to $C^{(g-1)-}$ in $\mathrm{Jac}(C)$ if and only if it has a g_k^1?

9. Generalizations of the Riemann-Hilbert problem In modern language, the Riemann-Hilbert problem can be described as follows. Each meromorphic $\mathrm{gl}(n, \mathbf{C})$-valued 1-form ω on $\mathbf{P}^1$, with at worst simple poles, is of the form

$$\omega = \sum_{k=1}^{N} \frac{A_k}{t - t_k}\, dt,$$

where each A_k is a constant $n \times n$ matrix and $t_k \in \mathbf{C}$. Set $D = \{t_1, \ldots, t_N, \infty\}$ and $V = \mathbf{P}^1 - D$. (Note that ∞ need only be included in D if $\mathrm{Res}_\infty \omega \neq 0$, i.e., $\sum A_k \neq 0$.) As in §2, ω defines a meromorphic connection on the trivial bundle $\mathbf{C}^n \times \mathbf{P}^1 \to \mathbf{P}^1$ with regular singular points along D. Since $d\omega + \omega \wedge \omega = 0$, the connection is flat and we have a monodromy representation

$$\rho \colon \pi_1(V) \to \mathrm{GL}(n, \mathbf{C}).$$

The Riemann-Hilbert (or Hilbert's 21st) problem asks if every linear representation $\pi_1(V) \to \mathrm{GL}(n)$ arises in this way. Lappo-Danilevsky [**19**], Plemelj [**24**], and others have shown that the answer is yes except in some degenerate cases. But the problem, as stated, remains unsolved (cf. [**29**, p. 311]).

An obvious generalization of the Riemann-Hilbert problem is the following. Suppose that V is a smooth variety and that X is a smooth completion of V such that $D = X - V$ is a divisor in X with normal crossings. A completely integrable 1-form $\omega \in \Omega^1(X \log D) \otimes \mathrm{gl}(n)$ defines a flat meromorphic connection on the trivial bundle $\mathbf{C}^n \times X \to X$ which has regular singularities along D. We will refer to the associated monodromy representation $\rho \colon \pi_1(V) \to \mathrm{GL}(n, \mathbf{C})$ as the monodromy of (the completely integrable 1-form) ω.

The first obvious question to ask is if every linear representation $\pi_1(V) \to \mathrm{GL}(n)$ is the monodromy representation of such an ω. The answer is no.

(9.1) EXAMPLE. Suppose that $\dim_{\mathbf{C}} \Omega^1(X \log D) = 1$. As we have seen previously, there is an isomorphism

$$(*) \qquad \Omega^1(X \log D) \to F^1 H^1(V; \mathbf{C}).$$

Choose a nonzero element w of $\Omega^1(X \log D)$. Every element ω of $\Omega^1(X \log D) \otimes \mathrm{gl}(n)$ is of the form $\omega = Aw$, where A is a constant $n \times n$ matrix. Note that

$d\omega = \omega \wedge \omega = 0$ so that all such ω are completely integrable. According to (2.5), the monodromy representation

$$\rho\colon \pi_1(V,x) \to \mathrm{GL}(n)$$

is given by

$$\begin{aligned}
\rho(\{\gamma\}) &= I + \int_\gamma \omega + \int_\gamma \omega\omega + \int_\gamma \omega\omega\omega + \cdots \\
&= I + \left(\int_\gamma w\right) A + \left(\int_\gamma ww\right) A^2 + \left(\int_\gamma www\right) A^3 + \cdots \\
&= I + \left(\int_\gamma w\right) A + \frac{1}{2!}\left(\int_\gamma w\right)^2 A^2 + \frac{1}{3!}\left(\int_\gamma w\right)^3 A^3 + \cdots \\
&= \exp\left(\int_\gamma wA\right) = \exp \int_\gamma \omega.
\end{aligned}$$

Here we have used the fact that if $w \in E^1(V)$, then

$$\int_\gamma \overbrace{ww\cdots w}^{r} = \frac{1}{r!}\left(\int_\gamma w\right)^r,$$

which can be proved using (2.11) or by direct calculation.

From (*) it follows that the dual of $\Omega^1(X \log D)$ is $H_1(V;\mathbf{C})/F^0H_1(V)$. Consequently, the monodromy representation ρ of ω factors through the 1-parameter subgroup of $\mathrm{GL}(n,\mathbf{C})$ generated by A:

$$\begin{tikzcd}
\pi_1(V) \arrow[r, "\theta"] \arrow[dr, "\rho"'] & H_1(V;\mathbf{C})/F^0 \cong \mathbf{C} \arrow[d, "\sigma_A"] \\
& \mathrm{GL}(n)
\end{tikzcd}$$

where $\theta(g) = (\int_g w)w^*$ and $\sigma_A(tw^*) = \exp(tA)$. Conversely, if a representation $\pi_1(V) \to \mathrm{GL}(n)$ factors as above, then it is the monodromy representation of Aw, where A is the infinitesimal generator of the 1-parameter subgroup σ_A.

We now consider two specific examples. In the first we will see how the Hodge filtration restricts the possible monodromy representations. Suppose that $V = \mathbf{C}/\Gamma$, an elliptic curve. Consider the representation

$$\begin{aligned}
\rho\colon \pi_1(V) &\to \mathrm{GL}(2,\mathbf{C}), \\
\gamma &\to \begin{pmatrix} 1 & \int_\gamma d\bar{z} \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Since the group $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$ is isomorphic to $\mathbf{C}$, ρ is a monodromy representation if and only if the functional

$$\begin{aligned}
\theta\colon \pi_1(V) &\to \mathbf{C}, \\
\gamma &\to \int_\gamma d\bar{z}
\end{aligned}$$

factors through $\pi_1(V) \to H_1(V)/F^0$. Since dz and $d\bar{z}$ are linearly independent in $H^1(V;\mathbf{C})$, θ does not factor and ρ is not a monodromy representation.

In the second example, the Hodge filtration is trivial. Suppose that V is the complement of the cusp $\{(x,y)\colon x^2 = y^3\}$ in $\mathbf{C}^2$. Its fundamental group is the group $\langle a,b\colon a^2 = b^3\rangle$. Viewing the symmetric group on three letters, Σ_3, as the set of permutation matrices in $\mathrm{GL}(3,\mathbf{C})$, we obtain a representation $\rho\colon \pi_1(V) \to \mathrm{GL}(3)$ by defining $\rho(a) = (1\ 2)$ and $\rho(b) = (1\ 2\ 3)$. Since the image of ρ is not abelian, ρ does not factor through the Hurewicz homomorphism $\pi_1(V) \to H_1(V)$. It follows that ρ is not the monodromy representation of a completely integrable 1-form $\omega \in \Omega^1(X \log D) \otimes \mathrm{gl}(3)$.

By now it should be clear that the correct generalization of the Riemann-Hilbert problem should be stated as follows.

(9.2) PROBLEM. Characterize the monodromy representations of completely integrable 1-forms $\omega \in \Omega^1(X \log D) \otimes \mathrm{gl}(n)$, where X is a smooth projective variety and D is a divisor in X with normal crossings.

This we have done in (9.1) when $\dim \Omega^1(X \log D) = 1$. When $\dim \Omega^1(X \log D) \geq 2$, we need a nonabelian analogue of $H_1(V)/F^0$. It turns out to be a quotient of the completed group ring of $\pi_1(V)$.

Suppose that G is a discrete group. As before, we shall denote the augmentation ideal of its group ring $\mathbf{C}G$ by J. Denote the J-adic completion of $\mathbf{C}G$ by $\mathbf{C}G\hat{}$. Note that if (V,x) is a pointed smooth quasi-projective variety, then by (4.1),

$$\begin{aligned}\mathbf{C}\pi_1(V,x)\hat{} &= \varprojlim \mathbf{C}\pi_1(V,x)/J^{s+1} = \varprojlim \operatorname{Hom}(H^0(B_s(V),x),\mathbf{C})\\ &= \operatorname{Hom}(\varinjlim H^0(B_s(V),x),\mathbf{C}) = \operatorname{Hom}(H^0(B(V),x),\mathbf{C}),\end{aligned}$$

where $H^0(B(V),x)$ is the union of the $H^0(B_s(V),x)$:

$$\begin{aligned}H^0(B(V),x) &= \bigcup H^0(B_s(V),x)\\ &= \{\text{iterated integrals that are homotopy functionals}\\ &\qquad\qquad \{\text{loops at } x\} \to \mathbf{C}\}.\end{aligned}$$

In this way $\mathbf{C}\pi_1(V,x)\hat{}$ acquires a Hodge filtration.

According to (2.9), the coproduct

$$\Delta\colon B(V) \to B(V) \otimes B(V),$$
$$\int w_1 \cdots w_r \to \sum_{i=0}^{r} \int w_1 \cdots w_i \otimes \int w_{i+1} \cdots w_r$$

is dual to the product of $\mathbf{C}\pi_1(V,x)$. Since the coproduct Δ preserves the Hodge and weight filtrations of $B(V)$, the product

$$\mathbf{C}\pi_1(V,x)\hat{} \otimes \mathbf{C}\pi_1(V,x)\hat{} \to \mathbf{C}\pi_1(V,x)\hat{}$$

preserves the Hodge filtration. Consequently the subspace

$$I = F^0 \cap J^1 + F^{-1} \cap J^2 + F^{-2} \cap J^3 + \cdots$$

of $\mathbf{C}\pi_1(V,x)\hat{}$ is an ideal. Set

$$A_\infty = \mathbf{C}\pi_1(V,x)\hat{}/I \quad \text{and} \quad A_s = \mathbf{C}\pi_1(V,x)\hat{}/I + J^{s+1}.$$

Denote the composite $\pi_1(V,x) \to \mathbf{C}\pi_1(V,x) \to A_s$ by $\theta_s(x)$, where $1 \le s \le \infty$. Note that $A_1 = \mathbf{C} \oplus H_1(V)/F^0$ and

$$\theta_1(x)\colon \pi_1(V,x) \to \mathbf{C} \oplus H_1(V)/F^0,$$
$$g \to (1, h(g)),$$

where h is the Hurewicz homomorphism. It is the homomorphisms $\theta_s(x)$ that generalize the homomorphism θ of (9.1).

(9.3) THEOREM [**30**]. *There exists a topological* $\mathbf{C}$*-algebra*

$$\mathcal{A} \subseteq \mathbf{C}\pi_1(V,x)\hat{}/I$$

such that

(a) $\operatorname{im}\theta \subseteq \mathcal{A}$,

(b) $\rho\colon \pi_1(V,x) \to \mathrm{GL}(n)$ *is a monodromy representation of an integrable* 1*-form on* V *with logarithmic singularities along* D *if and only if there exists a continuous* $\mathbf{C}$*-algebra homomorphism* $\varphi\colon \mathcal{A} \to \mathrm{GL}(n)$ *such that*

$$\begin{array}{ccc} \pi_1(V,x) & \xrightarrow{\rho} & \mathrm{GL}(n) \\ {\scriptstyle\theta}\downarrow & & \downarrow \\ \mathcal{A} & \xrightarrow[\varphi]{} & \mathrm{gl}(n) \end{array}$$

commutes. □

The characterization of monodromy representations given in (9.3) is not optimal, as it does not appear to help in solving the classical Riemann-Hilbert problem. Nonetheless, it still contributes nontrivial information when $X \neq \mathbf{P}^1$. For example, let R be the kernel of the natural map

$$\pi_1(V,x) \to \mathbf{C}\pi_1(V,x)\hat{}.$$

(9.4) COROLLARY. *If* ρ *is a monodromy representation, then* $\ker\rho \geq R$. □

This is a nontrivial restriction on monodromy representations as $\pi_1(V)/R \cong H_1(V)$ when $V = \mathbf{C}^2 - \{(x,y)\colon x^p = y^q\}$ with p,q relatively prime. Even when $\theta\colon \pi_1(V) \to \mathcal{A}$ is injective the theorem imposes restrictions on monodromy representations. For example, when V is an elliptic curve $\theta\colon \pi_1(V) \to \mathcal{A}$ is injective, but not every representation of $\pi_1(V)$ is a monodromy representation (cf. (9.1)).

(9.5) REMARK. The maps θ_s and θ_∞ were previously defined and studied by Parsin [**22**] and Hwang-Ma [**17**] in the case when V is a smooth projective curve of genus ≥ 2.

When $W_1H^1(V) = 0$ (e.g., V is a Zariski open subset of $\mathbf{P}^n$), $F^{-s}J^{s+1} = 0$ for all s. Consequently $I = 0$ and $A_\infty = \mathbf{C}\pi_1(V,x)\hat{}.$ Under the assumption that there exist $x_1, \ldots, x_l \in \pi_1(V,x)$ which are independent in $H_1(V;\mathbf{C})$ and generate $\operatorname{im}\theta$, it seems reasonable to make the following conjecture:

(9.6) CONJECTURE. *If V is a variety with $W_1H^1(V) = 0$ satisfying the conditions above, then $\rho\colon \pi_1(V,x) \to \mathrm{GL}(n)$ is the monodromy representation of a completely integrable 1-form $\omega \in \Omega^1(X \log D) \otimes \mathrm{gl}(n)$ if and only if ρ factors through $\operatorname{im}\theta$:*

$$\begin{tikzcd} \pi_1(V,x) \arrow[rr, "\theta"] \arrow[dr, "\rho"'] & & \operatorname{im}\theta \arrow[dl, dashed] \\ & \mathrm{GL}(n) & \end{tikzcd}$$ □

When $V = \mathbf{P}^1 - D$, then θ is injective. Thus this conjecture is a generalization of the classical Riemann-Hilbert problem. Golubeva [9] has solved the inverse problem when $V = \mathbf{P}^n -$ union of hyperplanes and the generators $\rho(x_j)$ of the monodromy group are sufficiently close to the identity. Her argument can be adapted to prove (9.6) for representations for which each $\rho(x_j) - I$ is sufficiently small. For nilpotent connections and unipotent representations we can completely characterize the monodromy representations.

A $\mathrm{gl}(n)$-valued 1-form ω is *nilpotent* if it takes values in a nilpotent subalgebra of $\mathrm{gl}(n)$.

(9.7) THEOREM [30]. *Suppose that $V = X - D$ is a smooth variety. A unipotent representation $\rho\colon \pi_1(V,x) \to \mathrm{GL}(m)$ is the monodromy representation of a completely integrable nilpotent 1-form $\omega \in \Omega^1(X \log D) \otimes \mathfrak{n}$ if and only if ρ factors through θ_m.*

$$\begin{array}{ccc} \pi_1(V,x) & \overset{\theta_m}{\to} & A_m \\ \downarrow & & \downarrow \\ \mathrm{GL}(m) & \to & \mathrm{gl}(m) \end{array}$$ □

When $W_1H^1(V) = 0$, $A_m = \mathbf{C}\pi_1(V,x)/J^{m+1}$. If $\rho\colon \pi_1(V,x) \to \mathrm{GL}(m)$ is unipotent, then ρ induces an algebra homomorphism

$$\hat{\rho}\colon \mathbf{C}\pi_1(V,x)/J^{m+1} \to \mathrm{gl}(m)$$

such that the diagram

$$\begin{array}{ccc} \pi_1(V,x) & \overset{\theta_m}{\to} & A_m \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\hat{\rho}} \\ \mathrm{GL}(m) & \to & \mathrm{gl}(m) \end{array}$$

commutes. Thus we have proved the next result.

(9.9) COROLLARY. *If $W_1H^1(V) = 0$, then every unipotent representation $\pi_1(V,x) \to \mathrm{GL}(m)$ is the monodromy representation of a completely integrable 1-form $\omega \in \Omega^1(X \log D) \otimes \mathrm{gl}(m)$.* □

When V is a Zariski open subset of $\mathbf{P}^m$, we recover a theorem of Aomoto [1].

Added in proof. In (7.3) the condition that

(a) $$u \wedge H^1_{\mathbf{Q}}(V) \to H^2_{\mathbf{Q}}(V)$$

not be injective is too strong. It should be replaced by the condition that

$$\text{(b)} \qquad E \hookrightarrow \bigwedge^2 H^1_{\mathbf{Q}}(V) \xrightarrow{p} H^1_{\mathbf{Q}}(V)/\mathbf{Q}u$$

be nontrivial, where p is interior multiplication by an element φ of $H_1(V)$ such that $\varphi(u) = 1$. Observe that if u satisfies (a), then it satisfies (b) so that if (a) is satisfied by all $u \in H^1_{\mathbf{Q}}(V)$, then Φ is finite to one.

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UNIVERSITY OF WASHINGTON