MATH 353.06, Quiz 8

Write your name here: ________________

November 30, 2012

1. (4pts) Write down the finite wave propagation problem, including the PDE and boundary/initial conditions.

Answer.

\[
\begin{cases}
    u_{tt} = a^2 u_{xx}, & 0 < x < L, \quad t > 0; \\
    u(0, t) = u(L, t) = 0, & t > 0; \\
    u(x, 0) = f(x), u_t(x, 0) = g(x), & 0 \leq x \leq L.
\end{cases}
\]

2. (4pts) Write down the infinite wave propagation problem, and the formula of its solution.

Answer.

\[
\begin{cases}
    u_{tt} = a^2 u_{xx}, & -\infty < x < \infty, \quad t > 0; \\
    u(x, 0) = f(x), u_t(x, 0) = g(x), & -\infty < x < \infty.
\end{cases}
\]

and

\[
u(x, t) = \frac{1}{2} (f(x + at) + f(x - at)) + \frac{1}{2a} \int_{x-at}^{x+at} g(y) \, dy.
\]
3. (7pts) Solve the following Laplace’s equation on a semi-infinite strip.

\[
\begin{aligned}
&u_{xx} + u_{yy} = 0, & 0 < x < L, y > 0; \\
&u(0, y) = u(L, y) = 0, & y \geq 0; \\
&u(x, 0) = h(x), \lim_{y \to \infty} u(x, y) \text{ is finite}, & 0 < x < L.
\end{aligned}
\]

**Answer.** Let

\[
\phi_n(x) = \sin \frac{n\pi}{L} x, \lambda_n = \frac{n^2 \pi^2}{L^2}, n \geq 1,
\]

and

\[
u(x, y) = \sum_{n=1}^{\infty} c_n(y) \phi_n(x).
\]

From the PDE we obtain that

\[-\lambda_n c_n(y) + c_n''(y) = 0, \quad n \geq 1.
\]

Its general solution is

\[
c_n(y) = \alpha_n e^{-\frac{n\pi}{L} y} + \beta_n e^{\frac{n\pi}{L} y}.
\]

In order for \(\lim_{y \to \infty} u(x, y)\) to be finite, we need \(\beta_n = 0\). Accordingly,

\[
u(x, y) = \sum_{n=1}^{\infty} \alpha_n e^{-\frac{n\pi}{L} y} \sin \frac{n\pi}{L} x.
\]

The initial condition implies that

\[
h(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi}{L} x,
\]

and in turn,

\[
\alpha_n = \frac{2}{L} \int_0^L h(x) \sin \frac{n\pi}{L} x \; dx.
\]