1. (6pts) Solve the following ODEs:

(a). \(x^2y'' - 3xy' + 5y = 0\), \(x > 0\)

**Solution.** This is an Euler equation with \(\alpha = -3, \beta = 5\). The characteristics equation is \(r^2 - 4r + 5 = 0\), which has two complex roots: \(r_{1,2} = 2 \pm i\). Thus, the general solution is \(y = c_1x^2\cos(\ln x) + c_2x^2\sin(\ln x)\).

(b). \(x^2y'' - xy' + y = 0\), \(x < 0\)

**Solution.** This is an Euler equation for negative \(x\). Let \(x = -e^t\). Then

\[
\begin{align*}
\frac{dx}{dt} &= -e^t = x \\
\frac{dy}{dt} &= \frac{dy}{dx} \frac{dx}{dt} = x \frac{dy}{dx} \\
\frac{d^2y}{dt^2} &= \frac{dx}{dt} \frac{dy}{dx} + x \frac{d}{dx} \left( \frac{dy}{dx} \right) \frac{dx}{dt} = x \frac{dy}{dx} + x^2 \frac{d^2y}{dx^2}.
\end{align*}
\]

Thus, the original ODE becomes \(\left( \frac{d^2y}{dx^2} - \frac{dy}{dx} \right) - \frac{dy}{dx} + y = 0\), or \(\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0\). The characteristics equation is \(r^2 - 2r + 1 = 0\), which has a repeated real root: \(r_1 = r_2 = 1\). Thus, the general solution is \(y = c_1e^t + c_2te^t = c_1(-x) + c_2(-x)\ln(-x)\).

Note that the Euler equation \(x^2y'' + \alpha xy' + \beta y = 0\), for \(x < 0\), still becomes \(\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0\), if one makes the substitution \(x = -e^t\). One can solve this ODE easily and set \(t = \ln(-x)\) to eliminate \(t\) and get a solution in terms of \(x\).
3. (9pts) Find a power series solution for the nonhomogeneous ODE near the given point:

\[ y'' - y = \frac{1}{1-x}, \quad x_0 = 0. \]

*(Hint: Use \( \frac{1}{1-x} = 1 + x + x^2 + \cdots \))*

**Solution.** Let \( y = \sum_{n=0}^{\infty} a_n x^n \), then

\[
\begin{align*}
y' &= \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m, \\
y'' &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = \sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m.
\end{align*}
\]

The ODE can be rewritten as

\[
\sum_{m=0}^{\infty} [(m+2)(m+1)a_{m+2} - a_m] x^m = \sum_{m=0}^{\infty} x^m.
\]

By comparing coefficients we get the following recurrence relation

\[
[(m+2)(m+1)a_{m+2} - a_m] = 1, \quad \text{for all} \quad m \geq 0,
\]

which gives

\[
\begin{align*}
m = 0: & \quad a_2 = \frac{1 + a_0}{2} \\
m = 1: & \quad a_3 = \frac{1 + a_1}{6} \\
m = 2: & \quad a_4 = \frac{1 + a_2}{12} = \frac{3 + a_0}{24} \\
m = 3: & \quad a_5 = \frac{1 + a_3}{20} = \frac{7 + a_1}{120} \\
& \vdots
\end{align*}
\]

The general solution is thus

\[
y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots \\
= a_0 + a_1 x + \left( \frac{1 + a_0}{2} \right) x^2 + \frac{1 + a_1}{6} x^3 + \frac{3 + a_0}{24} x^4 + \frac{7 + a_1}{120} x^5 + \cdots \\
= a_0 \left( 1 + \frac{1}{2} x^2 + \frac{1}{24} x^4 + \cdots \right) + a_1 \left( x + \frac{1}{6} x^3 + \frac{1}{120} x^5 + \cdots \right) \\
+ \left( \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{3}{24} x^4 + \frac{7}{120} x^5 + \cdots \right).
\]