1. (5pts) Interpret the following second-order nonhomogeneous ODE for $y = y(t)$ in the setting of a mass-spring system:

$$Ay'' + By' + Cy = f(t),$$

where $A, B, C$ are constants, $f(t)$ is a function (all known).

**Solution.** $y(t)$ represents the position of the particle at time $t$, $A$ the mass of the particle, $B$ damping coefficient, $C$ spring constant, $f(t)$ external force applied to the particle.

2. (10pts) Solve the following ODE or system of ODEs.

(a) 

$$9y'' - 6y' + y = 0$$

**Solution.** The characteristic equation is $9\lambda^2 - 6\lambda + 1 = 0$, or $(3\lambda - 1)^2 = 0$. There is a repeated root $\lambda = \frac{1}{3}$, thus the general solution of the ODE is $y(t) = (c_1 + c_2t)e^{\frac{t}{3}}$. 
(b)

\[
y_1' = 2y_1 + 4y_2 \\
y_2' = 3y_1 + y_2
\]

**Solution.** The system can be rewritten in matrix form

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.
\]

The general solution has the form

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2,
\]

where \( \lambda_1 \neq \lambda_2 \) are the eigenvalues of the coefficient matrix and \( v_1, v_2 \) the corresponding eigenvectors.

The characteristic polynomial of the matrix is

\[
\begin{vmatrix} 2 - \lambda & 4 \\ 3 & 1 - \lambda \end{vmatrix} = (2 - \lambda)(1 - \lambda) - 3 \cdot 4 = \lambda^2 - 3\lambda - 10.
\]

The eigenvalues are the roots of \( 0 = \lambda^2 - 3\lambda - 10 = (\lambda + 2)(\lambda - 5) \), that is, \( \lambda_1 = -2, \lambda_2 = 5 \).

To find \( v_1 \), we need to solve

\[
0 = \begin{pmatrix} 2 - \lambda_1 & 4 \\ 3 & 1 - \lambda_1 \end{pmatrix} v_1 = \begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} v_1,
\]

which yields \( v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \). Similarly, one can find \( v_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \).

Therefore, the general solution is

\[
\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 4 \\ 3 \end{pmatrix}.
\]