1 Terminology and Theory

A first-order, \(n\)-dimensional system is a union of \(n\) first-order differential equations in \(n\) unknowns

\[
\begin{align*}
x'_1 &= f_1(t, x_1, x_2, \ldots, x_n) \\
x'_2 &= f_2(t, x_1, x_2, \ldots, x_n) \\
\vdots \\
x'_n &= f_n(t, x_1, x_2, \ldots, x_n)
\end{align*}
\]

Using vector notation it can be conveniently represented as

\[
x' = f(t, x), \quad \text{where} \quad x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad f(t, x) = \begin{pmatrix} f_1(t, x) \\ f_2(t, x) \\ \vdots \\ f_n(t, x) \end{pmatrix}.
\]

It is said to be linear if it has the form

\[
x'(t) = \mathbf{A}(t)x(t) + \mathbf{b}(t),
\]

where \(\mathbf{A}(t) = (a_{ij}(t))\) is an \(n \times n\) matrix, \(\mathbf{b}(t) = (b_1(t), b_2(t), \ldots, b_n(t))^T\) is an \(n \times 1\) vector. The linear system is said to be homogeneous if \(\mathbf{b}(t) = \mathbf{0}\) for all \(t\).

The following theorem states that linear systems always have a solution. When an initial condition is specified, the solution is unique.

**Theorem 1.1.** Suppose \(a_{ij}(t), b_i(t)\) are all continuous on an interval \(I = (\alpha, \beta)\). Then for any \(t_0 \in I\) and \(\mathbf{x}_0 \in \mathbb{R}^n\), the initial value problem

\[
x'(t) = \mathbf{A}(t)x(t) + \mathbf{b}(t), \quad x(t_0) = \mathbf{x}_0
\]

has a unique solution defined for all \(t \in I\).

Homogeneous linear systems \(x'(t) = \mathbf{A}(t)x(t)\) enjoy the following properties:

- Any linear combination of solutions is also a solution.
- Any subset of solutions must be either always linearly independent or always linearly dependent.
- If the system has dimension \(n\), then any \(n\) linearly independent solutions span the general solution.
2 Solving Systems of ODEs

We focus on the following \( n \)-dimensional linear, homogeneous system

\[
\mathbf{x}'(t) = A \mathbf{x}(t),
\]

where \( A \) is an \( n \times n \) matrix with constant entries.

The following theorem indicates that solving the system is closely related to finding eigenvalues and eigenvectors of the matrix \( A \).

**Theorem 2.1.** Suppose that \( \lambda \) is an eigenvalue of the matrix \( A \), and \( \mathbf{v} \) is an associated eigenvector. Then \( \mathbf{x}(t) = e^{\lambda t} \mathbf{v} \) is a solution to the system \( \mathbf{x}' = A \mathbf{x} \).

### 2.1 Planar systems (\( n = 2 \))

Consider \( \mathbf{x}' = A \mathbf{x}, \ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). The characteristic polynomial is

\[
p(\lambda) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = \lambda^2 - T\lambda + D, \quad \text{with} \quad T = a + d, \ D = ad - bc.
\]

There are three cases about the roots of \( p(\lambda) = 0 \).

1. **Two distinct real roots** \( \lambda_1 \neq \lambda_2 \): Let \( \mathbf{v}_1, \mathbf{v}_2 \) be associated eigenvectors. The general solution is

\[
\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2.
\]

2. **Two complex roots** \( \lambda_{1,2} = \alpha \pm i\beta \): Let \( \mathbf{v}_1 \) be an eigenvector associated to \( \lambda_1 = \alpha + i\beta \). The general solution is

\[
\mathbf{x}(t) = c_1 \text{Re} \left( e^{\lambda_1 t} \mathbf{v}_1 \right) + c_2 \text{Im} \left( e^{\lambda_1 t} \mathbf{v}_1 \right).
\]

3. **One repeated real root** \( \lambda_1 = \lambda_2 \): The algebraic multiplicity of \( \lambda_1 \) is \( m_1 = 2 \), and there are two possible values of the geometric multiplicity \( d_1 \).

   - **\( d_1 = 2 \)** (The easy case): \( A = \lambda_1 \mathbf{I} \). The general solution is

     \[
     \mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_1 t} \mathbf{v}_2 = e^{\lambda_1 t} (c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2),
     \]

     where \( \mathbf{v}_1, \mathbf{v}_2 \) is any basis for \( \mathbb{R}^2 \).

   - **\( d_1 = 1 \)** (The interesting case): The general solution is

     \[
     \mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_1 t} (t \mathbf{v}_1 + \mathbf{v}_2),
     \]

     where \( \mathbf{v}_1, \mathbf{v}_2 \) satisfy

     \[
     (A - \lambda_1 \mathbf{I}) \mathbf{v}_1 = 0,
     (A - \lambda_1 \mathbf{I}) \mathbf{v}_2 = \mathbf{v}_1,
     \]

Finally, the above techniques also apply to 3-dimensional systems except the case of a single eigenvalue \( \lambda \) with \( m = 3, d = 1 \), which happens for example when \( A = \begin{pmatrix} \lambda & 1 \\ 1 & \lambda \\ \lambda & 1 \end{pmatrix} \). To solve the corresponding system, one may use the matrix exponential (see next page).
2.2 The matrix exponential

Let $A$ be an $n \times n$ matrix. Then for any $v \in \mathbb{R}^n$, the function $x(t) = e^{tA}v$ is a solution to

$$x' = Ax.$$ 

Therefore, the general solution is

$$x(t) = C_1e^{tA}v_1 + \cdots + C_ne^{tA}v_n$$

in which $v_1, \ldots, v_n$ is any basis for $\mathbb{R}^n$.

Recall that the exponential of a square matrix $M$ is defined to be

$$e^M = \sum_{k=0}^{\infty} \frac{1}{k!} M^k = I + M + \frac{1}{2}M^2 + \frac{1}{6}M^3 + \cdots.$$ 

The matrix exponential has the following properties.

- $e^O = I$.
- If $M$ is a nilpotent matrix of degree $\ell$, that is, $M^\ell = O$, then $e^M = I + M + \frac{1}{2}M^2 + \cdots + \frac{1}{(\ell-1)!}M^{\ell-1}$.
- If $M = \text{diag}(r_1, \ldots, r_n)$ is a diagonal matrix, then $e^M = \text{diag}(e^{r_1}, \ldots, e^{r_n})$. This implies that $e^{tI} = e^{tI}$.
- $e^{A+B} = e^A e^B$ if $A$ and $B$ commute, i.e., $AB = BA$. This implies that, for any square matrix $M$, the exponential $e^M$ is an invertible matrix whose inverse is $e^{-M}$.
- If $A^\ell v = 0$ for some $\ell \geq 1$, then

$$e^{tA}v = v + tAv + \cdots + \frac{t^{\ell-1}}{(\ell-1)!} A^{\ell-1}v.$$ 

In particular, if $\ell = 1$, then $e^{tA}v = v$; if $\ell = 2$, then $e^{tA}v = v + tAv$.

We use a basis of $\mathbb{R}^n$ consisting of eigenvectors and generalized eigenvectors of $A$. That is, each eigenvalue $\lambda$ should contribute $m$ basis vectors, where $m$ is its algebraic multiplicity, in the following order:

$$(A - \lambda I)v = 0 \quad (\text{eigenvectors})$$

$$(A - \lambda I)^2 v = 0 \quad (\text{generalized eigenvectors})$$

(\text{until } m \text{ linearly independent vectors have been found})

This basis facilitates the computing of $e^{tA}v$, due to the following result.

**Proposition 2.2.** If $(A - \lambda I)^\ell v = 0$ for some $\ell \geq 1$, then

$$e^{tA}v = e^{\lambda t}e^{t(A-\lambda I)}v = e^{\lambda t} \left( v + t(A - \lambda I)v + \cdots + \frac{t^{\ell-1}}{(\ell-1)!} (A - \lambda I)^{\ell-1}v \right).$$ 

In particular, if $\ell = 1$, then $e^{tA}v = e^{\lambda t}v$; if $\ell = 2$, $e^{tA}v = e^{\lambda t}(v + t(A - \lambda I)v)$. 
