1. Write down directly the formal solution to the following problems (including the equations determining the coefficients). Make sure that you understand each problem thoroughly and can quickly derive those solutions in an exam.

(a) \[
\begin{aligned}
& u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \; t > 0; \\
& u(x, 0) = f(x), \quad 0 \leq x \leq L; \\
& u(0, t) = u(L, t) = 0, \quad t > 0.
\end{aligned}
\]

(b) \[
\begin{aligned}
& u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \; t > 0; \\
& u(x, 0) = f(x), \quad 0 \leq x \leq L; \\
& u(0, t) = T_1, \; u(L, t) = T_2, \quad t > 0.
\end{aligned}
\]

(c) \[
\begin{aligned}
& u_t = \alpha^2 u_{xx}, \quad 0 < x < L, \; t > 0; \\
& u(x, 0) = f(x), \quad 0 \leq x \leq L; \\
& u_x(0, t) = u_x(L, t) = 0, \quad t > 0.
\end{aligned}
\]

(d) \[
\begin{aligned}
& u_{tt} = \alpha^2 u_{xx}, \quad 0 < x < L, \; t > 0; \\
& u(x, 0) = f(x), \; u_t(x, 0) = g(x), \quad 0 \leq x \leq L; \\
& u(0, t) = u(L, t) = 0, \quad t > 0.
\end{aligned}
\]

(e) \[
\begin{aligned}
& u_{xx} + u_{yy} = 0, \quad 0 < x < a, \; 0 < y < b; \\
& u(x, 0) = f(x), \; u(x, b) = 0, \quad 0 \leq x \leq a; \\
& u(0, y) = u(a, y) = 0, \quad 0 \leq y \leq b.
\end{aligned}
\]

(f) \[
\begin{aligned}
& u_{xx} + u_{yy} = 0, \quad 0 < x < a, \; 0 < y < b; \\
& u(x, 0) = 0, \; u(x, b) = g(x), \quad 0 \leq x \leq a; \\
& u(0, y) = u(a, y) = 0, \quad 0 \leq y \leq b.
\end{aligned}
\]

(g) \[
\begin{aligned}
& u_{xx} + u_{yy} = 0, \quad 0 < x < a, \; 0 < y < b; \\
& u(x, 0) = u(x, b) = 0, \quad 0 \leq x \leq a; \\
& u(0, y) = h(y), \; u(a, y) = 0, \quad 0 \leq y \leq b.
\end{aligned}
\]

(h) \[
\begin{aligned}
& u_{xx} + u_{yy} = 0, \quad 0 < x < a, \; 0 < y < b; \\
& u(x, 0) = u(x, b) = 0, \quad 0 \leq x \leq a; \\
& u(0, y) = 0, \; u(a, y) = k(y), \quad 0 \leq y \leq b.
\end{aligned}
\]

(i) \[
\begin{aligned}
& u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \quad 0 < r < a, \; -\pi < \theta < \pi; \\
& u(a, \theta) = f(\theta), \quad -\pi \leq \theta < \pi.
\end{aligned}
\]
2. Homework problems in Sections 10.5-8 (I recommend that you solve them again as if in an exam setting):

- 10.5: 7, 12
- 10.6: 8, 11, 12, 15
- 10.7: 4, 8, 9, 10
- 10.8: 8, 10

3. Consider the following Sturm-Liouville Boundary Value Problem:

\[ y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(1) + y(1) = 0 \]

1. Determine the eigenvalues/eigenfunctions of the problem;
2. Normalize the eigenfunctions;
3. Expand the function \( f(x) = 1 - x \) over the normalized eigenfunctions;
4. Solve the nonhomogeneous boundary value problem:

\[ y'' + 3y = x - 1, \quad y'(0) = 0, \quad y'(1) + y(1) = 0; \]

5. Use eigenfunction expansions to solve

\[ u_t = u_{xx} + (1-x)e^{-t}, \quad u_x(0,t) = 0, \quad u_x(1,t) + u(1,t) = 0, \quad u(x,0) = 0. \]

**Answers:**

1. Refer to textbook and lecture notes for formal solutions. We will compare solutions in review session.

2. Compare with your homework solutions.

3. (1) Eigenvalues \( \lambda_n = \mu_n^2, \) \( n = 1, 2, \ldots \) in which \( \mu_n \) satisfies \( \mu = \cot \mu; \)

   - eigenfunctions \( \phi_n(x) = \cos(\mu_n x). \)

   (2) Normalized eigenfunctions \( \phi_n(x) = k_n \cos(\mu_n x) \) with \( k_n^2 = \frac{2}{1 + \sin^2(\mu_n)}. \)

   (3) \( f(x) = \sum_{n=1}^{\infty} \frac{2(1-\cos \mu_n)}{\mu_n^2(1+\sin^2 \mu_n)} \cos(\mu_n x) \) or \( \sum_{n=1}^{\infty} \frac{2(1-\cos \mu_n)}{\mu_n^2 + \cos^2 \mu_n} \cos(\mu_n x). \)

   (4) \( y = \sum_{n=1}^{\infty} \frac{2(1-\cos \mu_n)}{\mu_n^2(\mu_n^2 - 3)(1+\sin^2 \mu_n)} \cos(\mu_n x). \)

   What if we have \( y'' + \lambda_1 y = x - 1 \) subject to same boundary conditions?

   (5) \( u(x, t) = \sum_{n=1}^{\infty} \frac{2(1-\cos \mu_n)}{\mu_n^2(\mu_n^2 - 1)(1+\sin^2 \mu_n)} (e^{-t} - e^{-\mu_n^2 t}) \cos(\mu_n x). \)

What about the following problem:

\[ u_t = u_{xx} + e^{-t}, \quad u_x(0,t) = 0, \quad u_x(1,t) + u(1,t) = 0, \quad u(x,0) = 1 - x. \]