1 General Concepts and Principles

One should understand the following

- **function**: a rule from one set of objects (domain) to another (target set)
- **field**: a function whose inputs are vectors
- **operator**: a function whose inputs are objects like vectors, or matrices, or even functions
- **linearity** of operators: An operator $L$ is said to be linear if $L(c_1f_1 + c_2f_2) = c_1L(f_1) + c_2L(f_2)$ for all scalars $c_1, c_2$ and objects $f_1, f_2$
- **vector space**: a space with two associated operations - addition and scalar multiplication, and closed under each.
- **principle of linear superposition**: If $L$ is a linear operator, then any linear combination of solutions of the equation $Lu = 0$ is also a solution of the equation. This implies that the null space of a linear operator $L$ (i.e., the set of all solutions of $Lu = 0$) is a vector space.
- **solutions of a nonhomogeneous equation** $Lu = f$ are of the form: a particular solution + general solutions of $Lu = 0$
- **eigenvalue problem for an operator** $L$: find all scalars $\lambda$ and associated objects $f \neq 0$ such that $L(f) = \lambda f$
- **initial- boundary value problem**: see the 1-D heat conduction problem as an example:

\[
\begin{align*}
\frac{\partial T}{\partial t} - a^2 \Delta T &= 0, \quad 0 < x < L, t > 0 \\
T(0, t) &= T(L, t) = 0, \quad t > 0 \\
T(x, 0) &= f(x), \quad 0 < x < L
\end{align*}
\]

(PDE) (boundary conditions) (initial condition)

2 First-order ODEs $\frac{dy}{dx} = f(x, y)$

Sometimes we also write $\frac{dy}{dt} = f(t, y)$ to emphasize that the dependent variable represents time. We have the following methods for studying first-order ODEs.
- Graphical method - direction field
- Numerical method - Euler’s method
  Given the ODE and an initial point \( y(x_0) = y_0 \), one can estimate \( y(x_n) \) for \( x_n = x_0 + nh \) by using the following recursive formula:
  \[ y(x_{i+1}) \approx y_i + h f(x_i, y_i) \]
- Analytic method for solvable first-order ODEs
  - Autonomous: \( \frac{dy}{dx} = f(y) \)
    - General solution: \( \int \frac{1}{f(y)} dy = t \)
    - One should know how to find equilibrium points/solutions, determine stability, and sketch some typical solutions
  - Separable: \( \frac{dy}{dx} = f(x)g(y) \) General solution is \( \int \frac{1}{f(x)} dx = \int f(x) dy \).
  - Homogeneous: \( \frac{dy}{dx} = f \) (RHS depends on \( x \) and \( y \) only through their ratio). To solve such an ODE, let \( u = y/x \), and rewrite it as \( u + x \frac{du}{dx} = f(u) \) which is separable \( \frac{du}{dx} = \frac{I(y) - u}{x} \).
  - Exact: \( M(x, y) + N(x, y) \frac{dy}{dx} = 0 \) with \( M_y = N_x \)
    - General solution is \( h(x, y) = C \), which is obtained by solving \( h_x = M, h_y = N \) together.
    - When the ODE is nonexact (i.e., \( M_y \neq N_x \)), an integrating factor may exist, for example,
      - If \( \frac{M_y - N_x}{N} \) only contains \( x \), then an integrating factor of the form \( \mu = \mu(x) \) exists: \( \frac{\mu'}{\mu} = \frac{M_y - N_x}{N} \)
      - If \( \frac{M_y - N_x}{M} \) only contains \( y \), then an integrating factor of the form \( \mu = \mu(y) \) exists: \( \frac{\mu'}{\mu} = -\frac{M_y - N_x}{M} \)
  - Linear: \( \frac{dy}{dx} + p(x)y = f(x) \)
    - Use the integrating factor \( \mu(x) = e^{\int p(x) dx} \) to transform the ODE into \( \frac{d}{dx}(y e^{\int p(x) dx}) = f(x) e^{\int p(x) dx} \)
    - The linear ODE with an arbitrary initial condition \( y(x_0) = y_0 \) has one and exactly one solution on any interval \( x \) such that \( x_0 \in I \) and \( p, f \) are both continuous.
  - System of first-order ODEs: \( \frac{dy}{dx} = Ay \), where \( y \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n} \).
    - Assuming that \( A \) has \( n \) linearly independent eigenvectors \( v_1, \ldots, v_n \)
      - (this is true for example when \( A \) is symmetric), the general solution is \( y(t) = c_1 v_1 e^{\lambda_1 t} + \ldots + c_n v_n e^{\lambda_n t} \), in which \( \lambda_1, \ldots, \lambda_n \) are corresponding eigenvalues.
  - Other ODEs that can be transformed to one of the above, e.g.,
    - Bernoulli equations \( y' + p(t)y = f(t)y^n \) (\( n \) is an integer)
      - If \( n = 0, 1 \), the equation is linear; for all other \( n \), let \( v = y^{1-n} \), then we obtain a linear first-order ODE \( \frac{dv}{1-n} + p(t)v = f(t) \)
3 Second-order ODEs

The most general form is

\[ P(x)y'' + Q(x)y' + R(x)y = F(x). \]  \hspace{1cm} (1)

When \( P(x) \neq 0 \), the ODE in (1) can be rewritten as

\[ y'' + p(x)y' + q(x)y = f(x), \]  \hspace{1cm} (2)

where

\[ p(x) = \frac{Q(x)}{P(x)}, q(x) = \frac{R(x)}{P(x)}, f(x) = \frac{F(x)}{P(x)}. \]

The ODE in (1) is said to be homogeneous if \( F(x) = 0 \); the ODE in (2) is said to be homogeneous if \( f(x) = 0 \).

3.1 Solving second-order ODEs by standard methods

In the following cases the ODE \( y'' + p(x)y' + q(x)y = f(x) \) is solvable.

- **Homogeneous** \( (f(x) = 0) \)
  
  - **Constant coefficients** \( y'' + py' + qy = 0 \) (\( p, q \) are numbers)
    The general solution is determined by the characteristic equation \( \lambda^2 + p\lambda + q = 0 \).
    
    * Two distinct real roots \( \lambda_1 \neq \lambda_2 \):
      
      \[ y(x) = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x} \]
    
    * Repeated real root \( \lambda_1 = \lambda_2 \):
      
      \[ y(x) = c_1e^{\lambda_1 x} + c_2xe^{\lambda_1 x} \]
    
    * Two complex roots \( \lambda_{1,2} = a \pm ib \):
      
      \[ y(x) = c_1e^{ax} \cos(bx) + c_2e^{ax} \sin(bx) \]
  
  - **Variable coefficients** \( y'' + p(x)y' + q(x)y = 0 \)
    If one already knows one solution \( y_1(x) \) (e.g., by observation), then the method of **reduction of order** can be used to find all solutions:
    
    \[ y(x) = v(x)y_1(x), \quad \text{with} \quad v'' = -\left(\frac{2y_1'}{y_1} + p\right)v'. \]

  - **Euler equations** \( x^2y'' + axy' + \beta y = 0 \) (for \( x > 0 \))
    Let \( x = e^t \), then the ODE becomes \( \frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + \beta y = 0. \)
    
    * Two distinct real roots \( \lambda_1 \neq \lambda_2 \):
      
      \[ y = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} = c_1x^{\lambda_1} + c_2x^{\lambda_2} \]
    
    * Repeated real root \( \lambda_1 = \lambda_2 \):
      
      \[ y = c_1e^{\lambda_1 t} + c_2te^{\lambda_1 t} = c_1x^{\lambda_1} + c_2x^{\lambda_1 \ln x} \]
Two complex roots $\lambda_{1,2} = a \pm ib$:

$$y = c_1 e^{at} \cos(bt) + c_2 e^{at} \sin(bt) = c_1 x^a \cos(b \ln x) + c_2 x^a \sin(b \ln x)$$

- **Nonhomogeneous** ($f(x) \neq 0$)

In the following we assume that we are given two linearly independent solutions $y_1(x), y_2(x)$ of the homogeneous ODE $y'' + p(x)y' + q(x)y = 0$. Recall that $y_1(x), y_2(x)$ are linearly independent iff their Wronskian is never zero:

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x) \neq 0.$$

It remains to find a particular solution $y_p$, by one of the following two methods:

- **Undetermined Coefficients**

  This method only works in the case of constant coefficients. One guesses a particular solution $y_p$ according to the specific form of the nonhomogeneous term $f(x)$ (see Section 3.5 of textbook on page 174).

- **Variation of Parameters**

  This method works with arbitrary functions $p(x), q(x)$, and yields a closed-form particular solution $y_p = \int_{x_0}^{x} G(x,s)f(s)ds$ in which $x_0$ is any conveniently picked number (e.g. $x_0 = 0$), and

$$G(x,s) = \begin{vmatrix} y_1(s) & y_2(s) \\ y_1'(s) & y_2'(s) \end{vmatrix} = \frac{\begin{vmatrix} y_1(s) & y_2(s) \\ y_1(x) & y_2(x) \end{vmatrix}}{W(y_1, y_2)(s)}.$$

The general solution of the nonhomogeneous ODE is $y = c_1 y_1 + c_2 y_2 + y_p$.

### 3.2 Interpretation as a mass-spring system

The second-order ODE with constant coefficients, $my'' + \epsilon y' + ky = f(t)$, can be interpreted as a mass-spring system. In this equation:

- $y = y(t)$ represents the position of the point particle at time $t$ ($y'(t), y''(t)$ then represent velocity and acceleration, respectively),
- $m$ is the mass of the particle,
- $\epsilon$ is the damping coefficient (modeling friction),
- $k$ is the spring constant (modeling spring force), and
- $f(t)$ is an external force acted on the particle.

One should understand the behavior of the solution to the ODE in the physical setting of a mass-spring system, and know what various terms mean (e.g., harmonic oscillator, amplitude, natural frequency, resonance).
3.3 Series solutions of \( P(x)y'' + Q(x)y' + R(x)y = 0 \)

In this part, one should first understand the following concepts:

- **power series function**: \( f(x) = \sum_{n\geq0} a_n(x-x_0)^n \). According to the **ratio test**, the series (absolutely) converges for all \( |x-x_0| < \rho \), with \( \rho = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| \) called the **radius of convergence**.

- **analytic at a point**: A function \( f(x) \) is said to be analytic at a point \( x_0 \) in its domain, if the function can be expanded into a power series at that point: \( f(x) = \sum_{n\geq0} a_n(x-x_0)^n \), with a positive radius of convergence. In fact, the coefficients must be \( a_n = \frac{f^{(n)}(x_0)}{n!} \) (this is called Taylor series).

- **analytic functions**: \( f(x) \) is said to be an analytic function if it is analytic everywhere in its domain. Note that all familiar functions are analytic at points where they are defined.

We assume that \( P, Q, R \) are all polynomials for simplicity (though analytic functions can be handled similarly), and can find series solutions for the ODE \( P(x)y'' + Q(x)y' + R(x)y = 0 \) near

- **an ordinary point** \( x_0 \): \( P(x_0) \neq 0 \). In this case, the series solution \( y(x) = \sum_{n\geq0} a_n(x-x_0)^n \) satisfies \( \rho(y) \geq \min(\rho(Q/P), \rho(R/P)) \). In fact, a nonhomogeneous ODE \( P(x)y'' + Q(x)y' + R(x)y = F(x) \), in which \( F \) is also a polynomial, or more generally, analytic at \( x_0 \), has a power series solution as well.

- **a regular singular point** \( x_0 \): \( P(x_0) = 0 \), but \( \lim_{x \to x_0} \frac{Q(x)}{P(x)}(x-x_0) \) and \( \lim_{x \to x_0} \frac{R(x)}{P(x)}(x-x_0)^2 \) both exist and are finite. The Euler equation \( x^2y'' + axy' + \beta y = 0 \) at point \( x_0 = 0 \) is such an example, which is solved by seeking solutions of the form \( y(x) = x^\gamma \).

3.4 Use of the Laplace Transform

The Laplace transform is a linear operator that acts on functions that are both **piecewise continuous** and of **exponential order**: 

\[
F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt
\]  

(3)

It can be used for solving the following initial value problem (IVP)

\[
ay'' + by' + cy = f(t), \quad y(0) = y_0, \quad y'(0) = y'_0,
\]

in which \( a, b, c, y_0, y'_0 \) are all constants. Let \( Y(s) = \mathcal{L}\{y(t)\} \) and \( F(s) = \mathcal{L}\{f(t)\} \). Taking the Laplace transform of the ODE yields that

\[
a(s^2Y(s) - sy_0 - y'_0) + b(sY(s) - y_0) + cY(s) = F(s),
\]

or

\[
Y(s) = \frac{(as + b)y_0 + ayy'_0}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}.
\]

Taking the inverse Laplace transform gives the desired solution to the IVP:

\[
y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{ \frac{(as + b)y_0 + ayy'_0}{as^2 + bs + c} \right\} + \mathcal{L}^{-1}\left\{ \frac{F(s)}{as^2 + bs + c} \right\}.
\]
Practice Problems

Disclaimer: This practice problems set is prepared by the instructor to test your understanding of the material in Chapters 1-3, 5 of the textbook and the Overview handout (pages 1-37). It is not meant to be comprehensive, nor to replace other study resources such as lecture notes, textbook, past quizzes, etc. Moreover, problems from this set or similar problems are not guaranteed to be on the exam. You should still use the review material together with other resources to fully prepare for the exam.

- Solve the following differential equations or initial value problems (make sure that you can identify the type of each ODE, possibly after rewriting the equation into a different form):
  1. \( y' + 2ty = 2t \)
  2. \( t^3y' + 4t^2y = e^{-t}, \quad y(-1) = 0 \) (and determine the largest interval on which the solution exists)
  3. \( y' = \frac{x^2}{x+1}, \quad y(0) = -2 \)
  4. \( y' = \frac{y}{x+1}, \quad y(1) = 1 \)
  5. \( y' = -4/y, \quad y(0) = y_0 \) (and determine how the interval in which the solution exists depends on the initial value \( y_0 \))
  6. \( (2xy^2 + 2y) + (2x^2y + 2x)y' = 1 \)
  7. \( (x + 2) \sin y \, dx + x \cos y \, dy = 0 \)
  8. \( 2y'' - 3y' + y = 0, \quad y(0) = 2, \quad y'(0) = 1/2 \) (and determine the maximum value of the solution)
  9. \( y'' + 2y' + 5y = 0, \quad y(0) = 2, \quad y'(0) = \alpha \) (and then find \( \alpha \) so that \( y = 0 \) when \( t = 1 \))
  10. \( y'' + y' + 0.25y = 0, \quad y(0) = 2, \quad y'(0) = b \) (and then determine the critical value of \( b \) separating solutions that grow positively from those that eventually grow negatively)
  11. \( t^2y'' - t(t+2)y' + (t+2)y = 0 \) (observe that \( g_1(t) = t \) is one solution)
  12. \( y'' + 4y = 2 \sin 3t \)
  13. \( y'' - 2y' + y = te^t + 4, \quad y(0) = y'(0) = 1 \)
  14. \( t^2y'' - t(t+2)y' + (t+2)y = 2t^3 \) (using results in problem 11 above)
  15. \( x^2y'' - xy' - 3y = 0 \) (\( x \neq 0 \))

- 16. Solve the ODE \( y'' + 4y = 3 \sin 2t \). The solution of this equation describes a special phenomenon of the mass-spring system. What is it?

- 17. Discuss the behavior of the solution of the ODE \( y'' + \epsilon y' + 4y = 0 \) when (1) \( \epsilon = 0 \), (2) \( 0 < \epsilon < 4 \), (3) \( \epsilon = 4 \), and (4) \( \epsilon > 4 \), in the setting of a mass-spring system.

- 18. Find approximate values of the solution of the following initial value problem at \( t = 0.1 \) and \( 0.2 \) using the Euler method with \( h = 0.1 \):

\[
y' = 2y - 1, \quad y(0) = 1.
\]
• 19. Determine and classify (as asymptotically stable or unstable) the equilibrium points, draw the phase line and sketch several graphs of solutions in the $ty$-plane, of the following equation:

$$\frac{dy}{dt} = -y(y - 1)(y - 2), \quad y(0) = y_0 \geq 0.$$ 

• 20. Find the radius of convergence of

(1) the power series $\sum_{n=0}^{\infty} \frac{2^n}{n!} (x - 1)^n$.

(2) the function $\frac{1}{\sqrt[3]{1 + x^2}}$ when expanded into a power series about $x_0 = 1$ (this is called Taylor series about $x_0 = 1$).

• 21. Consider the ODE

$$2x(x - 3)^2 y'' + 3xy' + (x - 3)y = 0.$$ 

(1) Find all its singular points and classify them as regular/irregular.

(2) Determine, without solving the ODE, a lower bound for the radius of convergence of the series solution about the ordinary point $x_0 = -1$. 

Answers to the practice problems

1. (Linear) $y = 1 + ce^{-t^2}$ ($c$ is an arbitrary constant, same below)

2. (Linear after diving each side by $t^3$) $y = -\frac{14t^4}{77}e^{-t}$, $(-\infty, 0)$

3. (Separable) $\frac{1}{2}y^2 = \frac{1}{3} \ln |1 + x^3| + 2$

4. (Homogeneous) $x/y = \ln |y| + 1$

5. (Autonomous) $y^2 = y_0^2 - 8t$, $(-\infty, y_0^2/8)$

6. (Exact) $x^2y^2 + 2xy = x + c$

7. (Can be made exact by multiplying $xe^x$) $x^2e^x \sin y = c$

8. (Second-order ODE with constant coefficients; next two the same) $y = 3e^{t/2} - e^t$, $y_{\text{max}} = \frac{9}{4}$

9. $y = e^{-t}(2 \cos 2t + (1 + \alpha/2) \sin 2t)$, $\alpha = -2 - 4 \cot 2$

10. $y = ((b - 1)t + 2)e^{t/2}$, $b = 1$

11. (Reduction of order, but first need to divide each side by $t^2$) $y = c_1 + c_2e^t$ (where $c_1, c_2$ are arbitrary constants, same below)

12. (Undetermined coefficients) $y = c_1 \cos 2t + c_2 \sin 2t - \frac{2}{5} \sin 3t$

13. (Undetermined coefficients) $y = (-3 + 4t + \frac{1}{6}t^3)e^t + 4$

14. (Variation of parameters. First need to divide out $t^2$) $y = c_1t + c_2te^t - 2t^2$

15. (Euler equation) General solution is $y = c_1 x^{-1} + c_2 x^3$ if $x > 0$ and $y = c_1(-x)^{-1} + c_2(-x)^3$ if $x < 0$. The two cases can be combined into one: $y = c_1 |x|^{-1} + c_2 |x|^3$ for all $x \neq 0$

16. (Undetermined coefficients) $y = c_1 \cos 2t + c_2 \sin 2t - \frac{2}{5} t \cos 2t$. Resonance.

17. (1) $\epsilon = 0$: $y = c_1 \cos (2t) + c_2 \sin (2t)$ (Harmonic oscillator)

(2) $0 < \epsilon < 4$: $y = c_1 e^{-\frac{\epsilon t}{2}} \cos \left(\frac{\sqrt{\epsilon^2 - 4} t}{2}\right) + c_2 e^{-\frac{\epsilon t}{2}} \sin \left(\frac{\sqrt{\epsilon^2 - 4} t}{2}\right)$ (Exponentially decaying with oscillation)

(3) $\epsilon = 4$: $y = e^{-\frac{t}{2}} (c_1 + c_2 t)$ (Exponentially decaying, but no oscillation)

(4) $\epsilon > 4$: $y = c_1 e^{-\frac{\epsilon t}{2} + \frac{\sqrt{\epsilon^2 - 4} t}{2}} + c_2 e^{-\frac{\epsilon t}{2} - \frac{\sqrt{\epsilon^2 - 4} t}{2}}$ (same as (3))

18. $y(0.1) \approx 1.1$, $y(0.2) \approx 1.22$


20. (1) $2$

(2) $\sqrt{3}$

21. (1) $0$ (regular singular), $3$ (irregular singular)

(2) $1$