Review Material for Sections 1.1-2.5

Math 104 Linear Algebra with Applications

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1 Vectors

For this part, one should be able to

- Calculate vector addition/subtraction, scalar multiplication, length, dot product and angle between vectors, projection of one vector onto another, and understand their geometric meanings
- Know (and prove!) the important identities and inequalities
- Use vector method to prove results in plane geometry
- Find parametric equations for lines and planes given
  - (for lines) a point and a direction, or two points
  - (for planes) a point and two directions, or three points
- Understand what hyperplanes are and find their point-normal equations
- Find intersections of lines, planes and hyperplanes
- Check whether a vector lies in the span of several other vectors: $\mathbf{v} \in \text{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ iff $[\mathbf{v}_1 \ldots \mathbf{v}_k] \mathbf{x} = \mathbf{v}$ has a solution

2 Matrix Algebra and Linear Transformations

For matrices one should know how to compute

- Row Echelon Form (REF) or reduced REF, and identify pivots
- Rank (by counting the number of pivots in the REF of the matrix)
- Standard matrix operations such as addition/subtraction, scalar multiplication, matrix multiplication, and transpose
- A sequence of elementary matrices, or their product, so that $\mathbf{E} \mathbf{A} = \mathbf{E}_k \cdots \mathbf{E}_1 \mathbf{A}$ is in (reduced) REF (for any given $\mathbf{A}$). Note that there are three types of elementary matrices $\mathbf{E}_{ij}, \mathbf{E}_i(c), \mathbf{E}_{ij}(c)$.

Two fast ways of finding the product $\mathbf{E}$:

- $[\mathbf{A}] \mathbf{I} \rightarrow [\mathbf{U}]\mathbf{E}$ via elementary row operations
- $[\mathbf{A}] \mathbf{b} \rightarrow [\mathbf{U}]\mathbf{E}\mathbf{b}$ via elementary row operations
• Inverse/right inverse/left inverse
  
  - When \( A \) is square and nonsingular: we can find the inverse by performing elementary row operations \( [A|I] \to [I|A^{-1}] \)
  
  - When \( A \in \mathcal{M}_{m\times n} \) is rectangular and \( \text{rank} A = m \): we may use \( [A|I_m] \to [(I_m,X)|E] \) to find right inverse
  
  - When \( A \in \mathcal{M}_{m\times n} \) is rectangular and \( \text{rank} A = n \): we may use \( [A|I_m] \to [(I_n,O)|E] \) to find left inverse

• The LU decomposition \( A_{m\times n} = L_{m\times m}U_{m\times n} \), in which \( L \) is lower-triangular with diagonal entries all equal to 1, and \( U \) is in (reduced) REF. Note that such a decomposition is not always possible.

Recall that there is a one-to-one correspondence between matrices \( A \in \mathcal{M}_{m\times n} \) and linear transformations \( T : \mathbb{R}^n \to \mathbb{R}^m \) in the following way:

\[
T = \mu_A \quad \text{for} \quad A = [T(e_1), T(e_2), \ldots, T(e_n)].
\]

There are three important linear transformations in \( \mathbb{R}^2 \):

- Projection (onto line \( \ell \)): \( P_\ell(x) = \frac{x \cdot a}{\|a\|^2} a = \frac{1}{\|a\|^2} a a^T x \)

- Reflection (about line \( \ell \)): \( R_\ell(x) = 2P_\ell(x) - x = \left( \frac{2}{\|a\|^2} a a^T - I \right) x \)

- Rotation (by angle \( \theta \)): \( \mu_{A_\theta}(x) = A_\theta x \), in which \( A_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \)

3 Systems of Linear Equations

Given a system of \( m \) linear equations in \( n \) unknowns

\[
\begin{align*}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}
\]

or written in short

\[
Ax = b,
\]

one can solve it in any of the following ways:

- Apply Gaussian elimination (by using elementary equation operations)
- Perform elementary row operations on the augmented matrix \([A|b]\)
- When \( A \) is square and nonsingular, use the inverse \( x = A^{-1}b \)

Note that \( Ax = b \) can only have zero, or one, or infinitely many solutions. We have the following existence and uniqueness results.

- \( Ax = b \) (for a particular \( b \)) has at least one solution iff \( \text{rank} A = \text{rank}[A|b] \) (interpreted as no contradicting row in \([A|b]\)).
• \( \mathbf{A} \mathbf{x} = \mathbf{b} \) has at least one solution for any \( \mathbf{b} \) iff \( \text{rank} \mathbf{A} = m \) (each row in the REF of \( \mathbf{A} \) contains a pivot, and thus one can use backward substitution to solve the system regardless of what \( \mathbf{b} \) is).

• \( \mathbf{A} \mathbf{x} = \mathbf{b} \) has at most one solution for any \( \mathbf{b} \) iff \( \text{rank} \mathbf{A} = n \) (each column contains a pivot, and thus there is no free variable). This is also equivalent to saying that the homogeneous system \( \mathbf{A} \mathbf{x} = \mathbf{0} \) has only the zero solution.

• \( \mathbf{A} \mathbf{x} = \mathbf{b} \) has a unique solution for any \( \mathbf{b} \) if \( \mathbf{A} \) is square and nonsingular.

The core of understanding the above is to analyze the REF of the augmented matrix \([\mathbf{A}|\mathbf{b}]\) (when \( \mathbf{b} \) is given) or coefficient matrix \( \mathbf{A} \) (when \( \mathbf{b} \) is arbitrary) and see if the REF contains or may contain a contradicting row, if there is a free variable, and whether one can use back-substitution to solve for each pivot variable.

**Practice Problems**

1. Given

\[
\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 1 & -2 \\ -3 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & -1 & 1 & 0 \\ -5 & 4 & -1 & -3 \\ 1 & -3 & 1 & 2 \end{pmatrix}.
\]

compute the following quantity (if it exists):

(a) \( \mathbf{B}^T \)

(b) \( \mathbf{B}^T \mathbf{A} \)

(c) \( \text{REF}(\mathbf{A}) \)

(d) \( \text{RREF}(\mathbf{B}) \)

(e) A left inverse for \( \mathbf{A} \)

(f) A right inverse for \( \mathbf{B} \)

(g) The LU decomposition for \( \mathbf{A} \)

2. Solve the following systems of linear equations

(a)

\[
x_1 + 2x_2 - x_3 + 2x_4 = 13 \\
2x_1 + 4x_2 - x_3 + 6x_4 = 19 \\
11x_1 + 22x_2 - 7x_3 + 30x_4 = 115
\]

(b)

\[
1x + 2y + 1z = -8 \\
-2y - 4z = 4 \\
2x + 4y = -12
\]

(c)

\[
2x + y = 5 \\
x - 3y = -1 \\
4x + 2y = 7
\]
3. Find the span of the column vectors of the following matrix
\[
A = \begin{pmatrix}
1 & -2 & 1 & -4 \\
2 & -1 & 8 & 1 \\
3 & -5 & 5 & -9
\end{pmatrix}.
\]

4. Write the matrix \( A \) below as a product of elementary matrices.
\[
A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix}.
\]
(Hint: Use elementary matrices to multiply \( A \) from the left to obtain the RREF, and then solve for \( A \).)

5. Determine whether the angle between the following two vectors is acute, right, or obtuse.
\[
u = \begin{pmatrix} 3 \\ -4 \\ 2 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 0 \\ 3 \\ -5 \end{pmatrix}.
\]

6. Find a parametric equation for the plane in \( \mathbb{R}^3 \) that contains two points (1, 2, 0), (2, 1, 3) and is parallel to the line described parametrically by
\[
x(t) = (3 - 4t, 2 + 3t, t - 2).
\]
What is the distance from the origin to the plane?

7. Let \( \mathcal{P} \) be the plane in \( \mathbb{R}^3 \) that contains the following points
\[
p = (0, 3, 2), \quad q = (3, 3, 1), \quad r = (2, 5, 0).
\]
   (a) Write down parametric equations for \( \mathcal{P} \).
   (b) Find the point-normal equation for \( \mathcal{P} \).

8. Determine the linear transformation \( T(x) \) that consists in first reflecting \( x \in \mathbb{R}^2 \) across the line \( x_1 = 0 \) and then projecting onto the line \( x_1 = x_2 \).

9. Suppose \( A \) is an \( m \times n \) matrix with rank 1. Prove that there are nonzero vectors \( u \in \mathbb{R}^m \) and \( v \in \mathbb{R}^n \) such that \( A = uv^T \). (Hint: Consider the REF \( EA = U \). Can you find vectors \( u_1 \) and \( v_1 \) such that \( U = u_1v_1^T \)?)

10. Suppose \( A \in \mathcal{M}_{m \times m} \) and \( B \in \mathcal{M}_{n \times n} \) are both invertible. Show that the following two \((m+n)\times(m+n)\) matrices are also invertible:
\[
\begin{pmatrix} A & O \\ O & B \end{pmatrix}, \quad \begin{pmatrix} A & C \\ O & B \end{pmatrix}
\]
(in the second matrix \( C \) is any \( m \times n \) matrix). To solve this problem, you need the result from homework problem 2.1.9 (block multiplication) on page 90. Then, for each of the two \((m+n)\times(m+n)\) matrices, guess the form of the inverse and then verify your guess is indeed the inverse.
Answers to the practice problems

1. (a) \( B^T = \begin{pmatrix} 0 & -5 & 1 \\ -1 & 4 & -3 \\ 1 & -1 & 1 \\ 0 & -3 & 2 \end{pmatrix} \)  
   (b) \( B^T A = \begin{pmatrix} -8 & 12 \\ 8 & -16 \\ 1 & 6 \\ -9 & 10 \end{pmatrix} \)  
   (c) \( \begin{pmatrix} 1 \\ -2 \\ 0 \\ 0 \end{pmatrix} \)  
   (d) \( \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \)  
   (e) \( \begin{pmatrix} 0 \\ -\frac{1}{4} \\ -\frac{1}{4} \\ -\frac{1}{4} \end{pmatrix} \)  
   (f) \( \frac{1}{7} \begin{pmatrix} 1 \\ -2 \\ -3 \\ 4 \\ -1 \\ -5 \\ 11 \\ -1 \\ -5 \end{pmatrix} \)  
   (g) \( \mathbf{L} = \begin{pmatrix} 1 \\ 0 \\ \frac{1}{2} \end{pmatrix}, \mathbf{U} = \begin{pmatrix} 5 \\ 0 \\ \frac{1}{2} \end{pmatrix} \)

2. (a) \( x_1 = 6 - 2x_2 - 4x_4, x_3 = -7 - 2x_4, \) and \( x_2, x_4 \) are free variables.  
   (b) \( x = -10, y = 2, z = -2 \)  
   (c) No solution

3. First observe that the problem is equivalent to finding all vectors \( \mathbf{b} = (b_1, b_2, b_3)^T \) so that \( A \mathbf{x} = \mathbf{b} \) is consistent. We form the augmented matrix

\[
\begin{pmatrix}
1 & -2 & 1 & -4 & | & b_1 \\
2 & -1 & 8 & 1 & | & b_2 \\
3 & -5 & 5 & -9 & | & b_3 \\
\end{pmatrix}
\]

and then apply elementary row operations to obtain

\[
\begin{pmatrix}
1 & -2 & 1 & -4 & | & b_1 \\
0 & 1 & 2 & 3 & | & b_3 - 3b_1 \\
0 & 0 & 0 & 0 & | & 7b_1 + b_2 - 3b_3 \\
\end{pmatrix}.
\]

In order for the equation to have a solution, we must have \( 7b_1 + b_2 - 3b_3 = 0 \).

The span of the column vectors thus is a plane in \( \mathbb{R}^3 \), with normal \( (7, 1, -3) \) and passing through the origin.

4. The elementary row operations performed below

\[
\begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 \\ 0 & -4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

correspond to left multiplication by elementary matrices

\[
\mathbf{E}_{21}(-3) \cdot \mathbf{E}_2(-\frac{1}{4}) \cdot \mathbf{E}_{12}(-2) \cdot \mathbf{A} = \mathbf{I}.
\]

Solving for \( \mathbf{A} \) we obtain

\[
\mathbf{A} = \left( \mathbf{E}_{21}(-3) \cdot \mathbf{E}_2(-\frac{1}{4}) \cdot \mathbf{E}_{12}(-2) \right)^{-1} = (\mathbf{E}_{12}(-2))^{-1} \cdot \left( \mathbf{E}_2(-\frac{1}{4}) \right)^{-1} \cdot (\mathbf{E}_{21}(-3))^{-1} = \mathbf{E}_{12}(2) \cdot \mathbf{E}_2(-4) \cdot \mathbf{E}_{21}(3).
\]
5. Since \( u \cdot v = 9 > 0 \), the angle between them is acute.

6. The direction of the line \((-4, 3, 1)\) is also a direction in the plane. Another direction in the plane is obtained by \((2, 1, 3) - (1, 2, 0) = (1, -1, 3)\). Thus, the parametric equation for the plane is

\[
x = (1, 2, 0) + s(1, -1, 3) + t(-4, 3, 1) = (1 + s - 4t, 2 - s + 3t, 3s + t),
\]

where \(s, t\) are parameters.

To find the distance from a point to the plane, we need to find a normal vector. One approach is to eliminate \(s, t\) in the parametric representation

\[
x_1 = 1 + s - 4t;
\]

\[
x_2 = 2 - s + 3t;
\]

\[
x_3 = 3s + t
\]

by solving the first two equations together for \(s, t\) and then plugging their formulas into the third equation. This yields a point-normal equation:

\[
10x_1 + 13x_2 + x_3 = 36.
\]

Thus, the distance is \(\frac{36}{\sqrt{10^2 + 13^2 + 1^2}} = \frac{36}{\sqrt{270}}\).

7. (a) \(x = (0, 3, 2) + s(3, 0, -1) + t(2, 2, -2) = (3s + 2t, 3 + 2t, 2s - 2t)\)

(b) One possibility is to use the method employed in the last problem.

Here, we adopt a different approach by solving the following system of linear equations to find a normal vector \(n\):

\[
\begin{pmatrix}
3 & 0 & -1 \\
2 & 2 & -2 \\
\end{pmatrix}
\begin{pmatrix}
n_1 \\
n_2 \\
n_3 \\
\end{pmatrix}
=
\begin{pmatrix}
0 \\
0 \\
\end{pmatrix}.
\]

Then \(n_1 = \frac{1}{3}n_3, n_2 = \frac{2}{3}n_3\) (\(n_3\) is a free variable). By setting \(n_3 = 3\) (or any other number), we find a normal vector \(n = (1, 2, 3)\). Thus, the point normal equation is \(x_1 + 2(x_2 - 3) + 3(x_3 - 2) = 0\).

8. \(T(x) = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} \\
\end{pmatrix}
\begin{pmatrix}
-1 & 0 \\
0 & 1 \\
\end{pmatrix}
x = \begin{pmatrix}
-\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} \\
\end{pmatrix}
x\)

9. \textbf{Proof.} There exists an invertible matrix \(E\) such that \(EA = U\) is in REF. Since \(\text{rank } A = 1\), only the first row of \(U\) is nonzero, i.e., \(U_2 = \cdots = U_m = 0\). Consequently, \(U = e_1U_1\), and thus \(A = E^{-1}U = (E^{-1}e_1) U_1\). Letting \(u = E^{-1}e_1\) and \(v = U_1^T\) completes the proof.

10. Verify directly that \(\begin{pmatrix}
A^{-1} & O \\
O & B^{-1} \\
\end{pmatrix}, \begin{pmatrix}
A^{-1} & -A^{-1}CB^{-1} \\
O & B^{-1} \\
\end{pmatrix}\) are their inverses.