1 Matrix Theory

For matrices one should know how to compute the following

- Row Echelon form or reduced REF, and identify pivots
- The rank (by counting the number of pivots in the REF of the matrix)
- Standard matrix operations such as addition/subtraction, scalar multiplication, matrix multiplication, and transpose
- The determinant of a square matrix using either of the following
  - by definition (expand along a row or column)
  - fast way for $2 \times 2$ and $3 \times 3$ matrices
  - fast way for diagonal, lower- and upper-triangular matrices (in this case, determinant = product of diagonal entries)
  - by “upper-triangularizing” the matrix (see practice problem 7)
- Inverse of a square nonsingular matrix $A$ using either of the following
  - $[A | I] \rightarrow [I | A^{-1}]$ via elementary row operations
  - Using the adjoint of $A$: $A^{-1} = 1/\det A \cdot \text{adj}A$

2 Solving Systems of Linear Equations

For a system of $m$ linear equations in $n$ unknowns

\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
    & \vdots \\
    a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{align*}

or written in short

\[ Ax = b, \]

one can solve it in any of the following ways:

- Apply Gaussian Elimination or Gauss-Jordan Elimination (by using elementary equation operations)
Perform elementary row operations on the augmented matrix \( \hat{A} \) (equivalent to above)

When \( A \) is square and nonsingular (there is only one solution then),

- use the inverse of \( A \): \( x = A^{-1} \cdot b \)
- use Cramer’s rule: \( x_i = \frac{\det(B_i)}{\det(A)}, i = 1, 2, \ldots, n \) (where \( B_i \) is the matrix obtained by replacing the \( i \)th column of \( A \) with \( b \))

3 Existence and Uniqueness of Solutions of Systems of Linear Equations \( Ax = b \)

One should understand thoroughly the following results. Fix an \( m \times n \) matrix \( A \). Then

- \( Ax = b \) (for a particular \( b \)) has at least one solution iff \( \text{rank}A = \text{rank}\hat{A} \) (interpreted as no contradicting row in \( \hat{A} \)).
- \( Ax = b \) always has at least one solution (for any \( b \)) iff \( \text{rank}A = m \) (each row in the REF of \( A \) contains a pivot, and thus one can use backward substitution to solve the system).
- \( Ax = b \) always has at most one solution (for any \( b \)) iff \( \text{rank}A = n \) (each column contains a pivot, and thus there is no free variable).
- \( Ax = b \) always has a unique solution (for any \( b \)) iff \( A \) is square and nonsingular, i.e., \( \text{rank}A = m = n \) (the RREF of \( A \) is the identity matrix).

The core of understanding the above is to analyze the REF of the augmented matrix (when \( b \) is given) or coefficient matrix (when \( b \) is arbitrary) and see if the REF contains or may contain a contradicting row, if there is a free variable, and whether one can use back-substitution to solve for each basic variable.

4 The Linear Implicit Function Theorem

Consider the system of linear equations (1) with \( m < n \).

- To determine whether a specific partition of the \( n \) variables, say \( \{x_1, \ldots, x_k\} \) and \( \{x_{k+1}, \ldots, x_n\} \), is a successful decomposition into endogenous and exogenous variables, one just needs to check whether the matrix

\[
A_k = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{pmatrix}
\]

is square (\( m = k \)) and nonsingular (\( \text{rank}A_k = m = k \)).

- To determine whether there exists a successful partition of endogenous and exogenous variables (without performing an exhaustive search), one just needs to examine the REF of the matrix \( A \). If \( \text{rank}A = m \), then the answer is yes and in fact, the basic variables and the free variables are such a partition.
5 Linear Algebra

For this part, one should be able to

• Calculate vector addition/subtraction, scalar multiplication, length and inner product, and understand their geometric meanings

• Write down parametric equations for lines and planes given
  – (for lines) a point and a direction, or two points
  – (for planes) a point and two directions, or three points

• Understand linear independence/dependence throughly and know how to check linear independence: $k$ vectors $v_1, \ldots, v_k$ in $\mathbb{R}^n$ are linearly independent if $k \leq n$ and one of the following is true
  – the equation $c_1v_1 + \cdots + c_kv_k = \mathbf{0}$ or in matrix notation $Ac = 0$, where $A = [v_1 \ v_2 \ \ldots \ v_k]$ and $c = (c_1, c_2, \ldots, c_k)^T$, has only the trivial solution $(0, \ldots, 0)$
  – $\text{rank} A = k$ (no free variables in the equation above)
  – (when $k = n$) $\det(A) \neq 0$ (nonsingular)

• Understand spanning sets and know how to determine the following
  – A vector $b \in \mathbb{R}^n$ lies in $L[v_1, \ldots, v_k]$ if the equation $Ac = b$ has at least one solution $c$ (this is true if $\text{rank} A = \text{rank} \hat{A}$)
  – $v_1, \ldots, v_k$ span $\mathbb{R}^n$ if $k \geq n$ and the equation $Ac = b$ has at least one solution $c$ for every vector $b \in \mathbb{R}^n$ (this is true if $\text{rank} A = n$)
  – (when $k = n$) $\det(A) \neq 0$

• Understand that a basis for a subset $V \subseteq \mathbb{R}^n$ is a set of $k$ vectors $v_1, \ldots, v_k$ satisfying the following
  – $L[v_1, \ldots, v_k] = V$
  – $v_1, \ldots, v_k$ are linearly independent

The number $k$ is referred to as the dimension of $V$. When $V$ is $\mathbb{R}^n$, then $k = n$, together with either of the above conditions, is enough to guarantee that $v_1, \ldots, v_k$ form a basis for $\mathbb{R}^n$. 
Practice Problems

1. Given
   \[ A = \begin{pmatrix} 5 & 2 & 0 \\ 1 & -5 & 0 \\ -3 & 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 & 1 & 0 \\ -5 & 4 & -1 & -3 \\ 1 & -3 & 1 & 2 \end{pmatrix}, \]
   compute the following quantities:
   (a) \( AB \)
   (b) \( B^T \)
   (c) \( \text{RREF}(B) \)
   (d) \( \text{rank}B \)
   (e) \( \text{det}A \)
   (f) \( \text{adj}A \)
   (g) \( A^{-1} \)
   (h) \( (A^T)^{-1} \)
   (i) \( \text{det}(A^T) \)
   (j) \( \text{det}(A^{-1}) \) (hint: apply \( \text{det} \) to both sides of \( AA^{-1} = I \))
   (k) \( \text{det}(A \cdot A \cdot A \cdot A) \)

2. Solve the following systems of linear equations
   (a)
   \[
   \begin{align*}
   x_1 + 2x_2 - x_3 + 2x_4 &= 13 \\
   2x_1 + 4x_2 - x_3 + 6x_4 &= 19 \\
   11x_1 + 22x_2 - 7x_3 + 30x_4 &= 115
   \end{align*}
   \]
   (b)
   \[
   \begin{align*}
   1x + 2y + 1z &= -8 \\
   -2y - 4z &= 4 \\
   2x + 4y &= -12
   \end{align*}
   \]
   (c)
   \[
   \begin{align*}
   2x + y &= 5 \\
   x - 3y &= -1 \\
   4x + 2y &= 7
   \end{align*}
   \]

3. Find all values of \((b_1, b_2, b_3)\) so that the following system have solutions.
   \[
   \begin{pmatrix} 1 & -2 & 1 & -4 \\ 2 & -1 & 8 & 1 \\ 3 & -5 & 5 & -9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}
   \]
   What can you say about the set of such vectors \((b_1, b_2, b_3)\)?
4. Write the matrix $A$ below as a product of elementary matrices.

$$ A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} $$

(hint: use elementary matrices to multiply $A$ from the left to obtain the RREF, and then solve for $A$.)

5. Bob says that the following system of equations can be used to view $T$ and $K$ as endogenous variables determined by the other variables. Is he correct? Explain why or why not.

$$ 4T + 4M + 2K = T + U $$
$$ 5T + z^2M + 5K = 2U - z $$

6. The system $Ax = b$ relates the variables $x_1, \ldots, x_6$. Unfortunately, the matrix $A$ has been corrupted so we do not know its entries. However, the reduced row echelon form of $A$ is recorded below:

$$ \text{rref}(A) = \begin{pmatrix} 1 & 2 & 0 & 6 & 4 & 0 \\ 0 & 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} $$

Based on the above information, determine which of the following sets of variables could be interpreted as endogenous in this system. (Make sure to justify each of your answers.)

(a) $\{x_1, x_3, x_4\}$
(b) $\{x_3, x_4, x_6\}$
(c) $\{x_4, x_5, x_6\}$

7. Let $A$ be an $n \times n$ matrix, and $B$ another $n \times n$ matrix, obtained by performing an elementary row operation on $A$. Show that $\det B = \det A$ iff the row operation is some $E_{ij}(r)$. Then use this result to compute the determinant of the following matrix.

$$ A = \begin{pmatrix} 1 & 2 & 0 & 6 \\ -2 & 1 & 2 & 3 \\ 1 & 3 & -2 & 3 \\ -1 & 1 & 4 & 0 \end{pmatrix} $$

(hint: to prove the result, write $B = EA$ with $E$ an elementary matrix; to compute the determinant, repeatedly apply the row operation to the given matrix until you reach the REF.)

8. Determine whether the angle between the following two vectors is acute, right, or obtuse.

$$ u = \begin{pmatrix} 3 \\ -4 \\ 2 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 0 \\ 3 \\ -5 \end{pmatrix} $$
9. Find a parametric equation for the plane in $\mathbb{R}^3$ that contains two points $(1, 2, 0), (2, 1, 3)$ and is parallel to the line described parametrically by

$$x(t) = (3 - 4t, 2 + 3t, t - 2).$$

10. Let $P$ be the plane in $\mathbb{R}^3$ that contains the following points $p = (0, 3, 2), q = (3, 3, 1), r = (2, 5, 0)$.

(a) Write down parametric equations for $P$.

(b) Find the point-normal equation for $P$

i. by using the cross product to find the normal vector;

ii. by letting $\mathbf{n} = (n_1, n_2, n_3)$ and solving the following system of linear equations to find a normal vector

$$(q - p) \cdot \mathbf{n} = 0$$

$$(r - p) \cdot \mathbf{n} = 0.$$ 

11. In each of the following, determine whether the set of vectors are linearly independent/dependent, whether they span $\mathbb{R}^3$, and whether they form a basis for $\mathbb{R}^3$.

(a) $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$;

(b) $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$;

(c) $v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 5 \\ 6 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 7 \\ 0 \end{pmatrix}$.

12. Let $A$ be an $n \times n$ square matrix, with $a_i$ representing its $i$th column. First explain why the following statements are equivalent:

- $A$ is nonsingular;
- $A$ is invertible;
- $A$ has a nonzero determinant.

Then, show that if $A$ is nonsingular, then its column vectors $a_1, \ldots, a_n$ form a basis for $\mathbb{R}^n$. 

6
Answers to the practice problems

1. (a) \[ \mathbf{AB} = \begin{pmatrix} -10 & 3 & 3 & -6 \\ 26 & -21 & 6 & 15 \\ -11 & 14 & -6 & -8 \end{pmatrix} \]

(b) \[ \mathbf{B}^T = \begin{pmatrix} 0 & -5 & 1 \\ -1 & 4 & -3 \\ 1 & -1 & 1 \\ 0 & -3 & 2 \end{pmatrix} \]

(c) \[ \text{RREF}(\mathbf{B}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \]

(d) \[ \text{rank} \mathbf{B} = 3 \]

(e) \[ \det \mathbf{A} = 27 \]

(f) \[ \text{adj} \mathbf{A} = \begin{pmatrix} 5 & 2 & 0 \\ 1 & -5 & 0 \\ -13 & -16 & -27 \end{pmatrix} \]

(g) \[ \mathbf{A}^{-1} = \frac{1}{27} \begin{pmatrix} 5 & 2 & 0 \\ 1 & -5 & 0 \\ -13 & -16 & -27 \end{pmatrix} \]

(h) \[ (\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T = \frac{1}{27} \begin{pmatrix} 5 & 1 & -13 \\ 2 & -5 & -16 \\ 0 & 0 & -27 \end{pmatrix} \]

(i) \[ \det (\mathbf{A}^T) = \det(\mathbf{A}) = 27 \]

(j) \[ \det (\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})} = \frac{1}{27} \]

(k) \[ \det(\mathbf{A} \cdot \mathbf{A} \cdot \mathbf{A} \cdot \mathbf{A}) = \det(\mathbf{A})^4 = 27^4 \]

2. (a) \[ x_1 = 6 - 2x_2 - 4x_4, \ x_3 = -7 - 2x_4, \text{ and } x_2, x_4 \text{ are free variables.} \]

(b) \[ x = -10, \ y = 2, z = -2 \]

(c) No solution

3. First form the augmented matrix

\[
\begin{pmatrix}
1 & -2 & 1 & -4 & | & b_1 \\
2 & -1 & 8 & 1 & | & b_2 \\
3 & -5 & 5 & -9 & | & b_3
\end{pmatrix}
\]

then apply elementary row operations to obtain

\[
\begin{pmatrix}
1 & -2 & 1 & -4 & | & b_1 \\
0 & 1 & 2 & 3 & | & b_3 - 3b_1 \\
0 & 0 & 0 & 0 & | & 7b_1 + b_2 - 3b_3
\end{pmatrix}
\]

In order for the equation to have a solution, we must have

\[ 7b_1 + b_2 - 3b_3 = 0. \]
The set of all such \((b_1, b_2, b_3)\) is actually a plane in \(\mathbb{R}^3\), with normal \((7, 1, -3)\) and passing through the point \((1, -4, 1)\). (Note that in order to get a point from the point-normal equation, one can set arbitrary values for two variables, for example \(b_1 = b_1 = 1\), then solve for the third \(b_2 = -4\). In fact, if we choose \(b_1 = b_2 = 0\), then \(b_3 = 0\), implying that the origin is also on the plane.)

4. The elementary row operations performed below
\[
\begin{pmatrix}
1 & 3 \\
2 & 2
\end{pmatrix} \to \begin{pmatrix}
1 & 3 \\
0 & -4
\end{pmatrix} \to \begin{pmatrix}
1 & 3 \\
0 & 1
\end{pmatrix} \to \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]
correspond to elementary matrices \(E_{12}(-2), E_2(-\frac{1}{4}), E_{21}(-3)\) (respectively), with
\[
E_{21}(-3) \cdot E_2(-\frac{1}{4}) \cdot E_{12}(-2) \cdot A = I.
\]
Solving for \(A\) we obtain
\[
A = \left( E_{21}(-3) \cdot E_2(-\frac{1}{4}) \cdot E_{12}(-2) \right)^{-1}
\]
\[
= (E_{12}(-2))^{-1} \cdot \left( E_2(-\frac{1}{4}) \right)^{-1} \cdot (E_{21}(-3))^{-1}
\]
\[
= E_{12}(2) \cdot E_2(-4) \cdot E_{21}(3).
\]

5. Bob is correct, because the coefficient matrix for the variables \(T, K, \begin{pmatrix} 4 & 2 \\ 5 & 5 \end{pmatrix}\)
is square and nonsingular (\(\det \neq 0\)).

6. Note that though \(A\) is not accessible, the new equations with \(rref(A)\) as coefficient matrix are equivalent to \(Ax = b\) (but the right hand of the new system is no longer \(b\). However, this is irrelevant for determining which variables are endogenous/exogenous).

(a) The corresponding coefficient matrix is \(\begin{pmatrix} 1 & 0 & 6 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}\). Since it is square but has a rank 2, the variables \(x_1, x_3, x_4\) can not be endogenous variables.

(b) Yes (using similar arguments above).

(c) No.

7. Write \(B = EA\) and take determinants on both sides: \(\det(B) = \det(E) \cdot \det(A)\).
Since we know \(\det(E_{ij}) = -1, \det(E_{ij}(r)) = r, \det(E_{ij}(r)) = 1\), \(E\) must be some \(E_{ij}(r)\) in order to have \(\det(B) = \det(A)\). Applying only row operations \(E_{ij}(r)\) to the given matrix, we obtain
\[
\det(A) = \det \begin{pmatrix}
1 & 2 & 0 & 6 \\
0 & 5 & 2 & 15 \\
0 & 0 & -\frac{12}{5} & -6 \\
0 & 0 & 0 & -10
\end{pmatrix}
= 1 \cdot 5 \cdot \left( -\frac{12}{5} \right) \cdot (-10) = 120.
\]
8. Since $\mathbf{u} \cdot \mathbf{v} = 9 > 0$, the angle between them is acute.

9. The direction of the line $(-4, 3, 1)$ is also a direction in the plane. Another direction in the plane is obtained by $(2, 1, 3) - (1, 2, 0) = (1, -1, 3)$. Thus, the parametric equation for the plane is
\[ \mathbf{x} = (1, 2, 0) + s(1, -1, 3) + t(-4, 3, 1) = (1 + s - 4t, 2 - s + 3t, 3s + t), \]
where $s, t$ are parameters.

10. (a) $\mathbf{x} = (0, 3, 2) + s(3, 0, -1) + t(2, 2, -2) = (3s + 2t, 3 + 2t, 2 - s - 2t)$
(b) Using cross product, one can obtain a normal vector to the plane $\mathcal{P}$ as follows:
\[
\mathbf{n} = \det \begin{pmatrix}
\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\
3 & 0 & -1 \\
2 & 2 & -2
\end{pmatrix} = 2\mathbf{e}_1 + 4\mathbf{e}_2 + 6\mathbf{e}_3 = (2, 4, 6).
\]
Thus the point normal equation is $2(x_1 - 0) + 4(x_2 - 3) + 6(x_3 - 2) = 0$
or $x_1 + 2x_2 + 3x_3 = 12$.
Alternatively, one can solve the following system of linear equations in order to find $\mathbf{n}$:
\[
\begin{pmatrix}
3 & 0 & -1 \\
2 & 2 & -2
\end{pmatrix}
\begin{pmatrix}
n_1 \\
n_2 \\
n_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]
Then $n_1 = \frac{1}{3}n_3$, $n_2 = \frac{2}{3}n_3$ ($n_3$ is a free variable). By setting $n_3 = 1$
(or any other number, e.g., $n_3 = 3$ which is a better choice leading to
all integer components of $\mathbf{n}$), we find a normal vector $\mathbf{n} = (\frac{1}{3}, \frac{2}{3}, 1)$.
Thus, the point normal equation is $\frac{1}{3}(x_1 - 0) + \frac{2}{3}(x_2 - 3) + 1(x_3 - 2) = 0$,
or equivalently $x_1 + 2x_2 + 3x_3 = 12$.

11. (a) Linearly independent; but cannot span $\mathbb{R}^3$; not a basis for $\mathbb{R}^3$
(b) Linearly dependent; can span $\mathbb{R}^3$; not a basis for $\mathbb{R}^3$
(c) Linearly independent; can span $\mathbb{R}^3$; thus a basis for $\mathbb{R}^3$

12. **Proof.** We prove the equivalence in a cyclic way (1 implies 2, 2 implies 3, and 3 then implies 1, thus 1,2,3 are all equivalent).

- **nonsingular implies invertible:** Since $\mathbf{A}$ is nonsingular (i.e. rank=n),
the equation $\mathbf{A}\mathbf{x} = \mathbf{e}_i$, for each $1 \leq i \leq n$, has a unique solution which
we denote by $\mathbf{c}_i$. Let $\mathbf{C} = [\mathbf{c}_1 \ldots \mathbf{c}_n]$, then $\mathbf{A}\mathbf{C} = [\mathbf{e}_1 \ldots \mathbf{e}_n] = \mathbf{I}$.
Thus, $\mathbf{C}$ is a right inverse of $\mathbf{A}$, which further implies that $\mathbf{A}$ is
invertible.

- **invertible implies nonzero determinant:** Let $\mathbf{C}$ be the inverse of $\mathbf{A}$, that is $\mathbf{A}\mathbf{C} = \mathbf{C}\mathbf{A} = \mathbf{I}$. Applying det to each part we have
$\det(\mathbf{A})\det(\mathbf{C}) = 1$, and therefore $\det(\mathbf{A}) \neq 0$.

- **nonzero determinant implies nonsingular:** Let $\mathbf{U}$ be the RREF of $\mathbf{A}$.
Then there exist elementary matrices $\mathbf{E}_1, \ldots, \mathbf{E}_k$ such that
\[
\mathbf{E}_k \cdots \mathbf{E}_1 \cdot \mathbf{A} = \mathbf{U}.
\]
Since $\det(\mathbf{E}_j) \neq 0$ for all $j = 1, \ldots, k$, thus $\det(\mathbf{U}) \neq 0$. This is true
only when $\mathbf{U} = \mathbf{I}$, so that rank$\mathbf{A} = n$, i.e. nonsingular.
If \( \mathbf{A} \) is nonsingular, then \( \mathbf{A} \mathbf{c} = \mathbf{0} \) has a unique solution \( \mathbf{c} = \mathbf{0} \), thus \( a_1, \ldots, a_n \) are linearly independent, which is enough to ensure that they form a basis for \( \mathbb{R}_n \).

Another proof is that, because \( \mathbf{A} \) is nonsingular, \( \mathbf{A} \mathbf{c} = \mathbf{b} \) for any \( \mathbf{b} \in \mathbb{R}_n \) always has a solution \( \mathbf{c} = \mathbf{A}^{-1} \mathbf{b} \). This shows that \( a_1, \ldots, a_n \) span \( \mathbb{R}_n \), and thus they can be a basis for \( \mathbb{R}_n \).