Unconstrained Optimization on Bounded and Closed Domains

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November 10, 2011

We consider the following optimization problem:

\[
\max / \min f(x) \quad \text{for} \quad x \in U,
\]

where \( U \subseteq \mathbb{R}^n \) is a set, and \( f : U \rightarrow \mathbb{R}^1 \) is a \( C^1 \) function.

If \( U \) is an open set (i.e., all points in \( U \) are interior points), then the global max/min of \( f \), if it exists, is found by the following two-step algorithm.

1. Find all critical points of the function \( f \) by solving \( \nabla f(x) = 0 \)
2. Determine the definiteness of the Hessian matrix \( D^2 f(x) \) on \( U \):
   - If \( D^2 f(x) \) is positive semidefinite for all \( x \in U \), then all critical points are global minimizers;
   - If \( D^2 f(x) \) is negative semidefinite for all \( x \in U \), then all critical points are global maximizers.

The above material is taught in class and also presented in the textbook. In this handout we discuss the case when the domain \( U \) is a bounded (i.e., it can be contained in a ball of finite radius) and closed (i.e., it contains its entire boundary) set.

When \( U \) is a bounded and closed subset of \( \mathbb{R}^n \), we can solve (1) by a three-step algorithm:

1. Find all local extrema of the function \( f \) inside \( U \) corresponding to interior critical points \( \nabla f(x) = 0 \)
2. Determine the global max/min of \( f \) on the boundary (an easier problem in general)
3. The max/min of the above values is the global max/min of the function \( f \) on the closed domain \( U \).

**Example.** Let \( f(x, y) = x(x^2 + y^2 - 1) \), and \( U = \{(x, y) \mid x^2 + y^2 \leq 1\} \), the closed unit disk in \( \mathbb{R}^2 \). Find the global max and min of the function \( f \) on \( U \).

**Solution:** We start by finding all critical points of \( f(x, y) = x^3 + xy^2 - x \) inside \( U \).

\[
\frac{\partial f}{\partial x} = 3x^2 + y^2 - 1 = 0 \\
\frac{\partial f}{\partial y} = 2xy = 0
\]
The second equation implies that either \( x = 0 \) or \( y = 0 \).

Case 1: If \( x = 0 \), then from the first equation we have \( y^2 - 1 = 0 \), hence \( y = \pm 1 \). However, these two points \((0, \pm 1)\) are not interior points and should be discarded.

Case 2: If \( y = 0 \), then from the first equation we have \( 3x^2 - 1 = 0 \), hence \( x = \pm \frac{1}{\sqrt{3}} \). We obtain two valid critical points \((\pm \frac{1}{\sqrt{3}}, 0)\).

The function values at the critical points are \( f(\frac{1}{\sqrt{3}}, 0) = -\frac{2}{3\sqrt{3}} \), and \( f(-\frac{1}{\sqrt{3}}, 0) = \frac{2}{3\sqrt{3}} \). On the boundary of \( U \), \( \{(x, y) \mid x^2 + y^2 = 1\} \), \( f \) is identically zero. Comparing these numbers we see that the global max/min of \( f \) is \( \frac{2}{3\sqrt{3}} \) and \( -\frac{2}{3\sqrt{3}} \), respectively, attained at the locations \((-\frac{1}{\sqrt{3}}, 0)\) and \((\frac{1}{\sqrt{3}}, 0)\).

Finally, the following theorem asserts that the max and min of a continuous function defined on a bounded and closed domain always exist.

**Theorem.** Let \( f : U \rightarrow \mathbb{R}^1 \) be a continuous function, where \( U \subseteq \mathbb{R}^n \) is bounded and closed. Then, there always exists two points \( x_1, x_2 \in U \) such that \( f(x_1) \) is the global maximum of the function \( f \) on its domain \( U \), and \( f(x_2) \) the global minimum.

However, if the domain \( U \) is open, then the global max/min of \( f \) on \( U \) does not necessarily exists, for example \( f(x, y) = \frac{1}{x^2 + y^2} \) and \( U = \{(x, y) \mid 0 < x^2 + y^2 < 1\} \).