EXAM 2
Math 353 ODE & PDE, Section 6
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Duke University, Fall 2012

You have 50 minutes.
No notes, no books, no calculators.

You must show ALL work and explain your reasoning CLEARLY
to receive full credit.

Please write your initials and ID number at the top of each page.

Good luck!

Name: ______________________  INSTRUCTOR’S SOLUTION

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1. ________  Signature: ______________________

2. ________  Date: ______________________

3. ________

4. ________

Total Score: ___________ (/100 points)
1. (40 pts) Consider the function space $L^2(0, 4)$.

(a) Write down the definition of this space.

**Answer.** \[ \{ f(x) \mid \int_0^4 f^2(x) \, dx < \infty \}, \text{ or } \{ \text{All square-integrable functions on } (0,4) \} \]

(b) Which of the following functions are in this space? Circle your answers.

\[
\frac{1}{x}, \quad x^3, \quad \cos\frac{\pi x}{2}, \quad \sin\frac{3\pi x}{4}, \quad e^x.
\]

(c) For which constant $k$ are the two functions $f(x) = k - x$ and $g(x) = x$ orthogonal in this space.

**Answer.** From

\[
0 = (k - x, x) = \int_0^4 (k - x)x \, dx = \frac{k}{2} x^2 - \frac{x^3}{3} \bigg|_0^4 = 8k - \frac{64}{3},
\]

we obtain that $k = \frac{8}{3}$.

(d) Write down an orthogonal basis for this space.

**Answer.** \{cos $\frac{m\pi}{4} x$\}$_{m \geq 0}$ is an OGB for $L^2(0, 4)$.

(You may also use \{sin $\frac{m\pi}{4} x$\}$_{m \geq 1}$ as an OGB for this space)

(e) For the orthogonal basis in (d), find their lengths in $L^2(0, 4)$ and use them to obtain an orthonormal basis for this space.

**Answer.** In $L^2(0, 4)$ we have

\[
\| 1 \|^2 = \int_0^4 1^2 \, dx = 4
\]

\[
\| \cos \frac{m\pi}{4} x \|^2 = \int_0^4 \cos^2 \frac{m\pi}{4} x \, dx = \frac{1}{2} \int_{-4}^4 \cos^2 \frac{m\pi}{4} x \, dx = \frac{1}{2} \cdot 4 = 2.
\]

Thus, \{ $\frac{1}{2}, \sqrt{2} \cos \frac{m\pi}{4} x$\}$_{m \geq 1}$ is an ONB for $L^2(0, 4)$.

If you used \{sin $\frac{m\pi}{4} x$\}$_{m \geq 1}$ in (d), then \{ $\frac{1}{\sqrt{2}} \cos \frac{m\pi}{4} x$\}$_{m \geq 1}$ is another ONB for $L^2(0, 4)$, because

\[
\| \sin \frac{m\pi}{4} x \|^2 = \int_0^4 \sin^2 \frac{m\pi}{4} x \, dx = \frac{1}{2} \int_{-4}^4 \sin^2 \frac{m\pi}{4} x \, dx = \frac{1}{2} \cdot 4 = 2.
\]
(f) Find the Fourier cosine series of the following function

\[ f(x) = 1 - \frac{x^2}{16}, \quad 0 \leq x < 4. \]

**Answer.** First note that \( L = 4 \). The Fourier coefficients are computed as follows

\[
a_0 = \frac{2}{4} \int_0^4 1 - \frac{x^2}{16} \, dx = \frac{1}{2} \left( x - \frac{x^3}{48} \right)]_0^4 = \frac{1}{2} \left( 4 - \frac{64}{48} \right) = \frac{4}{3};
\]

\[
a_m = \frac{2}{4} \int_0^4 \left( 1 - \frac{x^2}{16} \right) \cos \frac{m\pi}{4} x \, dx
\]

\[
= \frac{1}{2} \int_0^4 \cos \frac{m\pi}{4} x \, dx - \frac{1}{32} \int_0^4 x^2 \cos \frac{m\pi}{4} x \, dx
\]

\[
= 0 - \frac{1}{32} \cdot \frac{2 \cdot 4^3}{m^2 \pi^2} \cos m\pi
\]

\[
= -\frac{4}{m^2 \pi^2} \cos m\pi, \quad m \geq 1.
\]

Thus, the Fourier cosine series is

\[ f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos \frac{m\pi}{4} x = \frac{2}{3} - \sum_{m=1}^{\infty} \frac{4}{m^2 \pi^2} \cos m\pi \cos \frac{m\pi}{4} x. \]

(g) Use the series in (f) to show that

\[ \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \]

**Proof.** In the above series choose \( x = 4 \) and use \( f(4) = 0 \) to obtain

\[ 0 = \frac{2}{3} - \sum_{m=1}^{\infty} \frac{4}{m^2 \pi^2} \cos m\pi = \frac{2}{3} - \sum_{m=1}^{\infty} \frac{4}{m^2 \pi^2} \quad (\cos^2 m\pi = 1), \]

which is

\[ 0 = \frac{\pi^2}{6} - \sum_{m=1}^{\infty} \frac{1}{m^2}, \]

or

\[ \frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \]

If you choose \( x = 0 \) (and use \( f(0) = 1 \), you will obtain a different identity

\[ \frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \]
2. (20 pts) Find the eigenvalues and eigenfunctions of the following BVP

\[ y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(\pi) = 0. \]

**Answer.** We consider the following three cases separately:

(1) \( \lambda = 0 \). In this case, the general solution is \( y = c_1 x + c_2 \). The two boundary conditions imply that \( y(0) = c_2 = 0 \) and \( y'(\pi) = c_1 = 0 \). Thus, \( y = 0 \) and we conclude that \( \lambda = 0 \) is not an eigenvalue.

(2) \( \lambda > 0 \). Denote \( \mu = \sqrt{\lambda} \) for convenience. Then \( \lambda = \mu^2 \). The general solution in this case is \( y = c_1 \cos(\mu x) + c_2 \sin(\mu x) \). Its derivative is \( y' = -c_1 \mu \sin(\mu x) + c_2 \mu \cos(\mu x) \). The first boundary condition implies that \( y(0) = c_1 = 0 \). This together with the second boundary condition gives that \( y'(\pi) = c_2 \mu \cos(\mu \pi) = 0 \). In order for \( y \neq 0 \), \( c_2 \) cannot be zero (since \( c_1 \) is already zero). Hence, \( \cos(\mu \pi) = 0 \), or \( \mu \pi = \frac{2k-1}{2} \pi \) for any positive integer \( k \). Therefore, we obtain the following eigenvalues \( \lambda_k = \frac{(2k-1)^2}{4}, k \geq 1 \), with corresponding eigenfunctions \( \phi_k(x) = \sin \frac{2k-1}{2} x \).

(3) \( \lambda < 0 \). Denote \( \mu = \sqrt{-\lambda} \) for convenience. Then \( \lambda = -\mu^2 \). The general solution in this case is \( y = c_1 \cosh(\mu x) + c_2 \sinh(\mu x) \). Its derivative is \( y' = c_1 \mu \sinh(\mu x) + c_2 \mu \cosh(\mu x) \). The first boundary condition implies that \( y'(0) = c_1 = 0 \). This together with the second boundary condition gives that \( y'(\pi) = c_2 \mu \cosh(\mu \pi) = 0 \). Since \( \cosh(\mu \pi) \neq 0 \), we have \( c_2 = 0 \) as well. Thus, \( y = 0 \) and we conclude that any negative \( \lambda \) is not an eigenvalue.

Conclusion: The BVP has eigenvalues \( \lambda_k = \frac{(2k-1)^2}{4}, k \geq 1 \), and associated eigenfunctions \( \phi_k(x) = \sin \frac{2k-1}{2} x \).
3. (20 pts) Find the inverse Laplace transform of the following functions (your answers need to be explicit functions).

(a) \( F(s) = \frac{2s - 3}{s^2 + s - 6} \)

**Answer.** We write
\[
\frac{2s - 3}{s^2 + s - 6} = \frac{A}{s + 3} + \frac{B}{s - 2},
\]
which gives that
\[2s - 3 = A(s - 2) + B(s + 3).\]
By choosing \( s = -3, 2 \) we obtain \( A = \frac{9}{5}, B = \frac{1}{5} \) respectively. Therefore, the inverse is
\[f(t) = \frac{9}{5}e^{-3t} + \frac{1}{5}e^{2t}.\]

(b) \( G(s) = \frac{s}{(s^2 + 1)^2} \)

**Answer.** Observe that
\[G(s) = -\frac{1}{2}H'(s), \quad \text{with} \quad H(s) = \frac{1}{s^2 + 1}.\]
Since
\[
\mathcal{L}^{-1}\{H(s)\} = \sin t,
\]
\[
\mathcal{L}^{-1}\{H'(s)\} = (-t)\mathcal{L}^{-1}\{H(s)\} = (-t)\sin t,
\]
we have
\[g(t) = -\frac{1}{2} \cdot (-t) \sin t = \frac{1}{2}t \sin t.
\]
Alternatively, we can use convolution to get
\[g(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s^2 + 1} \cdot \frac{s}{s^2 + 1} \right\} = \sin t \ast \cos t = \int_0^t \sin(t - \tau) \cos \tau \, d\tau.
\]
Using the formula
\[
\sin A \cos B = \frac{\sin(A + B) + \sin(A - B)}{2},
\]
we get
\[g(t) = \frac{1}{2} \int_0^t \sin t + \sin(t - 2\tau) \, d\tau = \frac{1}{2} \left( \tau \sin t + \frac{\cos(t - 2\tau)}{2} \right) \bigg|_0^t = \frac{1}{2}t \sin t.
\]
4. (20 pts) Use the Laplace transform to solve the following initial value problem
\[ y'' - 4y' + 5y = f(t), \quad y(0) = 0, \quad y'(0) = 0 \]
where
\[ f(t) = \begin{cases} 5, & 5 \leq t < 20; \\ 0, & 0 \leq t < 5 \text{ and } t \geq 20. \end{cases} \]

**Answer.** First observe that
\[ f(t) = 5u_5(t) - 5u_{20}(t). \]
Applying the Laplace transform together with the given initial conditions yields that
\[ (s^2 - 4s + 5)Y(s) = \frac{5}{s}e^{-5s} - \frac{5}{s}e^{-20s}. \]
Solve for \( Y(s) \) to get
\[ Y(s) = \frac{5}{s(s^2 - 4s + 5)}e^{-5s} - \frac{5}{s(s^2 - 4s + 5)}e^{-20s} = H(s)e^{-5s} - H(s)e^{-20s}, \]
in which
\[ H(s) = \frac{5}{s(s^2 - 4s + 5)} = \frac{A}{s} + \frac{Bs + C}{s^2 - 4s + 5}. \]
Equating the numerators we obtain
\[ 5 = A(s^2 - 4s + 5) + s(Bs + C). \]
First set \( s = 0 \) and then compare coefficients to get
\[ 5 = 5A, \quad A + B = 0, \quad -4A + C = 0. \]
Thus,
\[ A = 1, \quad B = -1, \quad C = 4, \]
and correspondingly
\[ H(s) = \frac{1}{s} + \frac{-s + 4}{s^2 - 4s + 5} = \frac{1}{s} + \frac{-(s - 2) + 2}{(s - 2)^2 + 1} = \frac{1}{s} - \frac{s - 2}{(s - 2)^2 + 1} + \frac{2}{(s - 2)^2 + 1}. \]
It inverse is
\[ h(t) = 1 - e^{2t} \cos t + 2e^{2t} \sin t, \]
and the solution to the IVP is
\[ y(t) = u_5(t)h(t - 5) - u_{20}(t)h(t - 20). \]