**Theorem.** Suppose we have an equation

\[ G(x_1, x_2, y) = c \]

(c is a constant) and we know a solution point \((a_1, a_2, b)\). If

\[ \frac{\partial G}{\partial y}(a_1, a_2, b) \neq 0, \]

then the equation defines \(y\) as an implicit function of \(x_1, x_2\) near the given point \((a_1, a_2, b)\). Furthermore,

\[ \frac{\partial y}{\partial x_1}(a_1, a_2) = -\frac{\partial G}{\partial x_1}(a_1, a_2, b) \cdot \frac{\partial y}{\partial x_2}(a_1, a_2), \quad \frac{\partial y}{\partial x_2}(a_1, a_2) = -\frac{\partial G}{\partial x_2}(a_1, a_2, b) \cdot \frac{\partial y}{\partial y}(a_1, a_2, b). \]

**Informal proof.** Assuming \(y\) is a function of \(x_1, x_2\) near \((a_1, a_2, b)\), let us compute the partial derivatives \(\frac{\partial G}{\partial x_1}\) of the two sides of \(G(x_1, x_2, y) = c\). By the chain rule, we have

\[ \frac{\partial G}{\partial x_1} + \frac{\partial G}{\partial y} \frac{\partial y}{\partial x_1} = 0 \text{ near } (a_1, a_2, b). \]

In particular, at the point \((a_1, a_2, b)\) we have

\[ \frac{\partial G}{\partial x_1}(a_1, a_2, b) + \frac{\partial G}{\partial y}(a_1, a_2, b) \cdot \frac{\partial y}{\partial x_1}(a_1, a_2) = 0. \]

If

\[ \frac{\partial G}{\partial y}(a_1, a_2, b) \neq 0, \]

then we can solve for \(\frac{\partial y}{\partial x_1}(a_1, a_2)\) and obtain the first formula in the theorem. The other equation for \(\frac{\partial y}{\partial x_2}(a_1, a_2)\) can be derived similarly.

**Remarks.**

1. The case of more than one exogenous variable is very similar to the case of a single exogenous variable that we discussed in class: The condition is exactly the same \(\frac{\partial G}{\partial y}(a_1, a_2, b) \neq 0\). Since there are more than one exogenous variable now, we can only talk about and compute partial derivatives of \(y\).
2. Once we know the two partial derivatives \( \frac{\partial y}{\partial x_1}(a_1, a_2), \frac{\partial y}{\partial x_2}(a_1, a_2) \), we can then estimate \( y(a_1 + \Delta x_1, a_2 + \Delta x_2) \) by linear approximation:

\[
y(a_1 + \Delta x_1, a_2 + \Delta x_2) \approx b + \frac{\partial y}{\partial x_1}(a_1, a_2) \Delta x_1 + \frac{\partial y}{\partial x_2}(a_1, a_2) \Delta x_2.
\]

**Example.** Consider the equation \( G(x_1, x_2, y) = x_1^2 + x_2^2 + y^2 = 25 \), and a solution point \((0, 3, 4)\).

Since

\[
\frac{\partial G}{\partial y} = 2y = 8 \neq 0 \quad \text{at} \ (0, 3, 4),
\]

the equation defines \( y \) as an implicit function of \( x_1, x_2 \) near \((0, 3, 4)\). Also,

\[
\frac{\partial G}{\partial x_1} = 2x_1 = 0 \quad \text{at} \ (0, 3, 4),
\]

\[
\frac{\partial G}{\partial x_2} = 2x_2 = 6 \quad \text{at} \ (0, 3, 4).
\]

By the Implicit Function Theorem,

\[
\frac{\partial y}{\partial x_1}(0, 3) = \frac{0}{8} = 0, \quad \frac{\partial y}{\partial x_2}(0, 3) = \frac{-6}{8} = -\frac{3}{4}.
\]

We can also estimate the true value of \( y \) around \((0, 3)\), say at \((0.1, 2.8)\) (then \( \Delta x_1 = 0.1, \Delta x_2 = -0.2 \)), as follows:

\[
y(0.1, 2.8) \approx 4 + \frac{\partial y}{\partial x_1}(0, 3) \cdot 0.1 + \frac{\partial y}{\partial x_2}(0, 3) \cdot (-0.2)
\]

\[
= 4 + 0 + \left(-\frac{3}{4}\right) \cdot (-0.2)
\]

\[
= 4.15.
\]