EXAM 2
Math 102 Multivariable Calculus for Econ, Section 3
Instructor: Guangliang Chen
Duke University, Fall 2011
You have 75 minutes.
No notes, no books, no calculators.
You must show ALL work and explain your reasoning CLEARLY
to receive full credit.
Please write your initials and ID number at the top of each page.

Good luck!
Name: __________________________
ID number: ________________________

1. __________
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"I have adhered to the Duke Community Standard in completing this exam."
Signature: _________________________
Date: _____________________________

Total Score: __________ (/125 points)
1. (20 pts) Let $f(x) = \sqrt{x+y}$.

(a) Determine the (natural) domain, target space, and range of the function $f$.

- **domain**: $\{(x,y) : x+y \geq 0\}$
- **target**: $\mathbb{R}$
- **range**: $[0, \infty)$

(b) Sketch three level curves $f = 0, f = 1, f = 2$ in the same coordinate plane, and label all intercepts on the axes.

(c) Along which unit vector direction does the function increase the most quickly at the point $(2, 2)$?

\[
\nabla f = \left( \frac{1}{2\sqrt{x+y}}, \frac{1}{2\sqrt{x+y}} \right) = \left( \frac{1}{4}, \frac{1}{4} \right) \text{ at } (2, 2)
\]

\[
\vec{v} = \frac{\nabla f}{|\nabla f|} = \left( \frac{\frac{1}{4}}{\frac{1}{\sqrt{8}}} \right) = \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right)
\]
2. (10 pts) Rewrite the following functions in matrix form:

(a) \( f(x_1, x_2, x_3) = (x_1 + 2x_2 + 3x_3, x_1 - x_2, x_2 - x_3, x_3 - x_1) \)

\[
= \begin{pmatrix}
1 & 2 & 3 \\
1 & -1 & 0 \\
0 & 1 & -1 \\
-1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{pmatrix}
\]

(b) \( f(x_1, x_2, x_3) = -2x_2^2 + x_3^2 - 3x_1x_2 + 4x_1x_3 - 6x_2x_3 \)

\[
= \begin{pmatrix}
\lambda_1 & \lambda_2 & \lambda_3
\end{pmatrix}
\begin{pmatrix}
0 & -\frac{3}{2} & 2 \\
-\frac{3}{2} & -2 & -3 \\
2 & -3 & 1
\end{pmatrix}
\begin{pmatrix}
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{pmatrix}
\]
3. (15 pts) Use differentials to approximate $\sqrt{5.9^2 + 1.9^2 \cdot 3.9^2}$.

$$f(x, y, z) = \sqrt{x^2 + y^2 z^2}$$

$a = 6$, $\Delta x = -0.1$
$b = 2$, $\Delta y = -0.1$
$c = 4$, $\Delta z = -0.1$

$$f(6, 2, 4) = \sqrt{6^2 + 2^2 \cdot 4^2} = 10$$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 z^2}} = \frac{6}{10}$$

$$\frac{\partial f}{\partial y} = \frac{yz^2}{\sqrt{x^2 + y^2 z^2}} = \frac{32}{10}$$

$$\frac{\partial f}{\partial z} = \frac{y^2 z}{\sqrt{x^2 + y^2 z^2}} = \frac{16}{10}$$

at $(6, 2, 4)$

Thus,

$$f(5.9, 1.9, 3.9) \approx f(6, 2, 4) +$$

$$\frac{\partial f}{\partial x}(6, 2, 4) \Delta x + \frac{\partial f}{\partial y}(6, 2, 4) \Delta y + \frac{\partial f}{\partial z}(6, 2, 4) \Delta z$$

$$= 10 + \frac{6}{10} (-0.1) + \frac{32}{10} (-0.1) + \frac{16}{10} (-0.1)$$

$$= 10 - 0.54$$

$$= 9.46$$
4. (15 pts) Suppose an ice cylinder is melting under the sun. At the moment when the
radius of the cylinder is \( r = 10 \text{ cm} \) and the height of the cylinder is \( h = 50 \text{ cm} \), the
radius is decreasing at a rate of \( 0.1 \text{ cm/min} \) and the height is decreasing at a rate of \( 0.25 \text{ cm/min} \). At what rate is the volume of the ice cylinder changing?
(Hint: the volume formula for the cylinder is \( V = \pi r^2 h \).)

When \( r = 10 \text{ cm}, \ h = 50 \text{ cm} \), we know

\[
\frac{dr}{dt} = -0.1 \text{ cm/min}
\]

\[
\frac{dh}{dt} = -0.25 \text{ cm/min}.
\]

Thus, by chain rule + product rule,

\[
\frac{dV}{dt} = \frac{d}{dt}(\pi r^2) \cdot h + \pi r^2 \frac{dh}{dt}
\]

\[
= 2\pi r \frac{dr}{dt} \cdot h + \pi r^2 \frac{dh}{dt}
\]

\[
= 2\pi \cdot 10 \cdot (-0.1) \cdot 50 + \pi \cdot 10^2 \cdot (-0.25)
\]

\[
= -125\pi \text{ cm}^3/\text{min}
\]

That is, the volume is decreasing at a rate of

\( 125\pi \text{ cm}^3/\text{min} \).
5. (15 pts) Given that \( F(u, v) = (u + v, v^2) \) and \( u = x^2 + y^2, v = xy \), compute the Jacobian derivative matrix of \( F \), viewed as a function of \( x, y \), at the point \( (x, y) = (1, 1) \).

\[
\begin{pmatrix}
\frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\
\frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\
\frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v}
\end{pmatrix} \begin{pmatrix}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 \\
0 & 2v
\end{pmatrix} \begin{pmatrix}
2x & 2y \\
y & x
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 \\
0 & 2
\end{pmatrix} \begin{pmatrix}
2 & 2 \\
1 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
3 & 3 \\
2 & 2
\end{pmatrix}
\]

when \( x = y = 1 \),

\( u = 2, \ v = 1 \)

\[
\begin{pmatrix}
3 & 3 \\
2 & 2
\end{pmatrix}
\]

at \( (x, y) = (1, 1) \).
6. (15 pts) Consider the following equation

\[ f(x, y, z) = x^2 y - y^2 z - xz^2 = -19. \]

(a) Verify that \( x = 1, y = 3, z = 2 \) is a regular point of the level set.

\[
\nabla f(1, 3, 2) = \left( 2xy - z^2, x^2 - 2yz, -y^2 - 2xz \right)_{x=1, y=3, z=2} = (2, -11, -13)
\]

(b) Which variable(s) can be viewed as an implicit function of the other two variables locally around \((1, 3, 2)\)?

all variables

(c) Estimate \( z \) when \( x = 1.05, y = 2.95 \) for the given equation.

\[
\frac{\partial^2}{\partial x} (1, 3) = -\frac{\frac{\partial F}{\partial x}(1, 3, 2)}{\frac{\partial F}{\partial z}(1, 3, 2)} = \frac{2}{13}
\]

\[
\frac{\partial^2}{\partial y} (1, 3) = -\frac{\frac{\partial F}{\partial y}(1, 3, 2)}{\frac{\partial F}{\partial z}(1, 3, 2)} = -\frac{11}{13}
\]

Thus, \( z \approx 2 + \frac{2}{13} (0.05) + (-\frac{11}{13})(-0.05) \)

\[
= 2 + \frac{0.65}{13} = 2.05.
\]
7. (20 pts) Consider the system of two equations in three unknowns

\[ G_1(x, y, z) = x + 2y + z = 5 \]
\[ G_2(x, y, z) = 3x^2yz = 12 \]

(a) Show that at the point \( x = 2, y = 1, z = 1 \), we can treat \( z \) as an exogenous variable and \( x, y \) as endogenous variables.

\[
\begin{pmatrix}
\frac{\partial G_1}{\partial x} & \frac{\partial G_1}{\partial y} \\
\frac{\partial G_2}{\partial x} & \frac{\partial G_2}{\partial y}
\end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 6xyz & 3x^2z \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 12 & 12 \end{pmatrix}
\]

Since \( \text{det} \begin{pmatrix} 1 & 2 \\ 12 & 12 \end{pmatrix} = -12 \neq 0 \),

\( x, y \) can be treated as endogenous variables near the point \( (2, 1, 1) \).

(b) If \( z \) rises to 1.2, use the Implicit Function Theorem to estimate the corresponding \( x \) and \( y \).

\[
\begin{pmatrix}
\frac{\partial x}{\partial z} (1) \\
\frac{\partial y}{\partial z} (1)
\end{pmatrix} = - \begin{pmatrix}
\frac{\partial G_1}{\partial x} (2,1,1) & \frac{\partial G_1}{\partial y} (2,1,1) \\
\frac{\partial G_2}{\partial x} (2,1,1) & \frac{\partial G_2}{\partial y} (2,1,1)
\end{pmatrix}^{-1} \begin{pmatrix}
\frac{\partial G_1}{\partial z} (2,1,1) \\
\frac{\partial G_2}{\partial z} (2,1,1)
\end{pmatrix}
\]

\[
= - \begin{pmatrix} 1 & 2 \\ 12 & 12 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 12 \end{pmatrix}
\]

\[
= \frac{1}{12} \begin{pmatrix} 12 & -2 \\ -12 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 12 \end{pmatrix}
\]

\[
= \begin{pmatrix} -1 \\ 0 \end{pmatrix}
\]

When \( z = 1.2 \), i.e., \( \Delta z = 0.2 \),

\[ x \approx 2 + (-1) \cdot 0.2 = 1.8 \]

\[ y \approx 1 + 0 \cdot 0.2 = 1 \]
8. (15 pts) Consider the function $F: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with

$$F(x, y) = (x^2 - y^2, 2xy).$$

(a) At which points is $F$ locally invertible?

$$DF(x, y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

$$\text{det} \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix} = 4(x^2 + y^2) \neq 0 \text{ for all } (x, y) \text{ except } (0, 0)$$

Therefore, $F$ is locally invertible everywhere in $\mathbb{R}^2$ except at the origin $(0, 0)$.

(b) Is $F$ globally one-to-one? Please justify your answer in either case.

No, because $F(-x, -y) = F(x, y)$ for any $(x, y)$

For example, $F(1, 1) = F(-1, -1) = (0, 2)$. 