EXAM 2
Math 104 Linear Algebra with Applications, Section 2
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Duke University, Spring 2012

You have 75 minutes.
No notes, no books, no calculators.
You must show ALL work and explain your reasoning CLEARLY
to receive full credit.

Good luck!

Name: Instructor’s Solution
ID number: ____________________

1. __________
2. __________
3. __________ Signature: ____________________
4. __________ Date: ____________________
5. __________
6. __________
7. __________ Total Score: __________ (/100 points)

“I have adhered to the Duke Community Standard in completing this exam.”
1. (15 pts) Let \( U, V \subset \mathbb{R}^3 \) be two arbitrary subspaces. Decide whether the following sets are also subspaces. *No matter your answer is yes or no, give an example supporting your answer in each part.*

(a) \( U \cap V \)

Yes. For example, \( U = x\text{-axis}, V = y\text{-axis} \), then \( U \cap V = \{0\} \), still a subspace.

(b) \( U \cup V \)

No. For example, \( U = x\text{-axis}, V = y\text{-axis} \), then \( U \cup V \) is not closed under vector addition.

(c) \( U + V \)

Yes. For example, \( U = x\text{-axis}, V = y\text{-axis} \), then \( U + V = xy\text{-plane} \), a subspace.

(d) \( U^\perp \)

Yes. For example, \( U = x\text{-axis} \), then \( U^\perp = yz\text{-plane} \), a subspace.

(e) \( (V^\perp)^\perp \)

Yes. For example, \( V = y\text{-axis} \), then \( V^\perp = xz\text{-plane} \), \( (V^\perp)^\perp = y\text{-axis} \), still a subspace.
2. (20 pts) Let

\[ A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & -1 & -3 \end{pmatrix}. \]

Find a basis for each of the four fundamental subspaces associated to \( A \).

Solution: Apply elementary row operations to obtain that \( EA = U \) in reduced REF form, with

\[ U = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}. \]

Then, each subspace has basis:

- \( R(A) \) : \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \end{pmatrix};
- \( C(A) \) : \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix};
- \( N(A) \) : \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \end{pmatrix};
- \( N(A^T) \) : \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix}. 

3. (10 pts) Let

\[ \mathbf{A} = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 2 & 2 & -2 \end{pmatrix}. \]

(a) Give constraint equations for \( \mathbf{C}(\mathbf{A}) \). What is the dimension of \( \mathbf{C}(\mathbf{A}) \)?

Solution: Apply row operations to obtain that

\[
\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix} \]

Thus, there is one constraint equation for \( \mathbf{C}(\mathbf{A}) \): \(-b_1 + b_2 + b_3 = 0\). The column space has dimension 2 because it is a plane in \( \mathbb{R}^3 \).

(b) Find a subset of the column vectors \( \{ \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \} \) that gives a basis for \( \mathbf{C}(\mathbf{A}) \). Please justify your answer.

Solution: From part (a) we know that \( \dim \mathbf{C}(\mathbf{A}) = 2 \). Hence, any two linearly independent column vectors of \( \mathbf{A} \) form a basis for \( \mathbf{C}(\mathbf{A}) \). Since every pair of column vectors are linearly independent, each of the three subsets of 2 column vectors form a basis.
4. (10 pts) Let \( A \) be an \( m \times n \) matrix, and \( B \) an \( n \times p \) matrix. Prove that

(a) \( \text{rank}(AB) \leq \text{rank}(A) \)

Proof: We prove that \( C(AB) \subset C(A) \). Let \( b \in C(AB) \), then \( b = (AB)x \) for some \( x \in \mathbb{R}^p \). Writing \( b = A(Bx) \), this shows that \( b \in C(A) \). Therefore, \( \dim C(AB) \leq \dim C(A) \), that is \( \text{rank}(AB) \leq \text{rank}(A) \).

(b) \( \text{rank}(AB) \leq \text{rank}(B) \)

Proof: We prove that \( N(B) \subset N(AB) \). Let \( x \in N(B) \), then \( Bx = 0 \). Multiplying each side by \( A \) from left gives that \( ABx = 0 \). This shows that \( x \in N(AB) \). Therefore, \( \dim N(B) \leq \dim N(AB) \), that is \( p - \text{rank}(B) \leq p - \text{rank}(AB) \). Clearly, \( \text{rank}(AB) \leq \text{rank}(B) \).

Alternatively, you may prove this inequality by establishing that \( R(AB) \subset R(B) \): Let \( b = x(AB) \in R(AB) \), where \( b, x \) are considered as row vectors (in order for the multiplication to be well defined). Then \( b = (xA)B \in R(B) \). Therefore, \( \text{rank}(AB) = \dim R(AB) \leq \dim R(B) = \text{rank}(B) \).
5. (15 pts) Let $V = \text{Span}((1, -1, 0, 2), (1, 0, 1, 1)) \subset \mathbb{R}^4$, and let $b = (1, -3, 1, 1)$.

(a) Find an orthogonal basis for $V$.

Solution: Let $v_1 = (1, -1, 0, 2), v_2 = (1, 0, 1, 1)$. Since $v_1, v_2$ are linearly independent, they form a basis for $V$. We can then use the Gram-Schmidt process to obtain an orthogonal basis:

$$w_1 = v_1 = (1, -1, 0, 2);$$

$$w'_2 = v_2 - \frac{v_2 \cdot w_1}{\|w_1\|^2}w_1 = \frac{1}{2}(1, 1, 2, 0).$$

Note that we may replace $w'_2$ with $w_2 = 2w'_2 = (1, 1, 2, 0)$ in order to have integer coordinates. The pair \{$w_1, w_2$\} is still an orthogonal basis for $V$.

(b) Find $p = \text{proj}_V b$.

Solution: $p = \text{proj}_{w_1} b + \text{proj}_{w_2} b = \frac{6}{6}(1, -1, 0, 2) + \frac{0}{6}(1, -3, 1, 1) = (1, -1, 0, 2)$. 


(c) Find the projection matrix onto \( V^\perp \).

Solution: The projection matrix onto \( V \) is

\[
P_V = P_{w_1} + P_{w_2} = \frac{1}{\|w_1\|^2}w_1^Tw_1 + \frac{1}{\|w_2\|^2}w_2^Tw_2
\]

\[
= \frac{1}{6} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} (1 \\ -1 \\ 0 \\ 2) + \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} (1 \\ 1 \\ 2 \\ 0)
\]

\[
= \frac{1}{3} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{pmatrix}.
\]

Therefore, the projection matrix onto \( V^\perp \) is

\[
P_{V^\perp} = I_4 - P_V = \frac{1}{3} \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & 1 \\ -1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix}.
\]
Let $\ell$ be the line in $\mathbb{R}^2$ passing through the origin and along the vector $a = (1, 2)$.

(a) Find a vector $b \in \mathbb{R}^2$ such that $B = \{a, b\}$ is an orthogonal basis for $\mathbb{R}^2$.

Solution: $b = (-2, 1)$.

(b) Find $[\text{proj}_\ell]_B$, the matrix for the projection onto $\ell$ with respect to $B$.

Solution: Treating $a, b$ as column vectors, we have

$$ (\text{proj}_\ell a \text{ proj}_\ell b) = (a 0) = (a \ b) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. $$

Thus, $[\text{proj}_\ell]_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

(c) Use change-of-basis formula to compute the standard matrix $[\text{proj}_\ell]_{\text{stand}}$.

Solution: Let $P = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}$ whose columns are $a, b$. Then,

$$ [\text{proj}_\ell]_{\text{stand}} = P [\text{proj}_\ell]_B P^{-1} $$
$$ = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}^{-1} $$
$$ = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}. $$

Note that your answer must be equal to $P_a = \frac{1}{\|a\|^2} a^T a$, which provides a way for knowing whether your answer is indeed correct.
7. (15 pts) Suppose $A$ is an $n \times n$ matrix with the property that $\text{rank}(A) = \text{rank}(A^2)$.

(a) Show that $N(A^2) = N(A)$.

**Proof:** We prove that $N(A) \subset N(A^2)$. Let $x \in N(A)$, that is $Ax = 0$. Clearly, $A^2x = AAx = A0 = 0$. This shows that $x \in N(A^2)$. Note that $\dim N(A) = n - \text{rank}(A) = n - \text{rank}(A^2) = \dim N(A^2)$. Therefore, $N(A^2) = N(A)$.

(b) Prove that $C(A) \cap N(A) = \{0\}$.

**Proof:** Let $x \in C(A) \cap N(A)$. Our goal is to show that $x = 0$. Since $x \in C(A)$, $x = Ay$ for some $y \in \mathbb{R}^n$. Also, $x \in N(A)$ implies that $Ax = 0$. Combining the two equations yields that $A^2y = 0$. This shows that $y \in N(A^2)$. Using part (a), we conclude that $y \in N(A)$. That is, $0 = Ay = x$.

(c) We also have $R(A) \cap N(A) = \{0\}$, which is true for any matrix $A$. Based on this fact and part (b), can we conclude that $R(A) = C(A)$? *If your answer is yes, please prove it; otherwise, give a counterexample.*

**Solution:** We cannot conclude this in general. One counterexample is the following: $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Since $A$ is square and $A = A^2$, the given conditions are satisfied. It is clear that $C(A)$ and $R(A)$ are different.