EXAM 1
Math 102 Multivariable Calculus for Econ, Section 3
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Duke University, Fall 2011

You have 75 minutes.

No notes, no books, no calculators.

You must show ALL work and explain your reasoning CLEARLY to receive full credit.

Please write your initials and ID number at the top of each page.

Good luck!

Name: __________________________
ID number: ______________________

1. __________
2. __________
3. __________
4. __________
5. __________
6. __________
7. __________

"I have adhered to the Duke Community Standard in completing this exam."

Signature: ______________________
Date: __________________________

Total Score: __________ (/125 points)
1. (15 pts) Solve the following system of linear equations, by the indicated method.

(a) Augmented matrix.

\[
\begin{pmatrix}
2 & 3 & -1 & | & 5 \\
1 & 1 & 1 & | & -1 \\
-1 & 2 & 3 & | & -4 \\
0 & -1 & 1 & | & -3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 1 & | & -1 \\
1 & 1 & 1 & | & 5 \\
2 & 3 & -1 & | & -4 \\
-1 & 2 & 3 & | & -3
\end{pmatrix}
\]

\[
\rightarrow
\begin{pmatrix}
1 & 1 & 1 & | & -1 \\
0 & 1 & -3 & | & 7 \\
0 & 3 & 4 & | & -5 \\
0 & -1 & 1 & | & -3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & 1 & | & -1 \\
0 & 1 & -3 & | & 7 \\
0 & 0 & 13 & | & -26 \\
0 & 0 & -2 & | & 4
\end{pmatrix}
\]

\[
\rightarrow
\begin{pmatrix}
1 & 1 & 1 & | & -1 \\
0 & 1 & -3 & | & 7 \\
0 & 0 & 1 & | & -2 \\
0 & 0 & 0 & | & 0
\end{pmatrix}
\rightarrow
\begin{cases}
\alpha_1 + \alpha_2 + \alpha_3 = -1 \\
\alpha_2 - 3\alpha_3 = 7 \\
\alpha_3 = -2
\end{cases}
\]

\[
\begin{cases}
\alpha_1 = -1 - \alpha_2 - \alpha_3 = 0 \\
\alpha_2 = 3\alpha_3 + 7 = 1 \\
\alpha_3 = -2
\end{cases}
\]

(b) Cramer’s Rule.

\[
\alpha = \frac{\begin{vmatrix}
-3 & -5 \\
11 & 4
\end{vmatrix}}{\begin{vmatrix}
2 & -5 \\
7 & 4
\end{vmatrix}} = \frac{43}{43} = 1
\]

\[
y = \frac{\begin{vmatrix}
2 & -3 \\
7 & 11
\end{vmatrix}}{\begin{vmatrix}
2 & -5 \\
7 & 4
\end{vmatrix}} = \frac{43}{43} = 1
\]
2. (30 pts) Let

\[ A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix}, \quad x = (2, -3, 0), \quad y = (1, 1, 1). \]

Compute the following quantities.

(a) The distance between the two points \( x \) and \( y \).

\[
\sqrt{(2-1)^2 + (-3-1)^2 + (0-1)^2} = \sqrt{1 + 16 + 1} = \sqrt{18} = 3\sqrt{2}.
\]

(b) The angle between the two vectors \( x \) and \( y \).

\[
\cos \theta = \frac{\vec{x} \cdot \vec{y}}{\| \vec{x} \| \cdot \| \vec{y} \|} = \frac{2 \cdot 1 + (-3) \cdot 1 + 0 \cdot 1}{\sqrt{2^2 + (-3)^2 + 0^2} \sqrt{1^2 + 1^2 + 1^2}} = \frac{-1}{\sqrt{13} \sqrt{3}} = \frac{-1}{\sqrt{39}}.
\]

\[ \theta = \arccos \left( -\frac{1}{\sqrt{39}} \right) \]

(c) \( x^T y \) (where \( x, y \) are viewed as row matrices).

\[
\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ -3 & -3 & -3 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[ 3 \times 1 \quad 1 \times 3 \quad 3 \times 3 \]
(d) $A^2$

$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 3 & 1 \\ 1 & 4 & 3 \\ 4 & 10 & 7 \end{pmatrix}$

(e) $\det(A)$

$2 \begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} - 1 \cdot \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + 0$

$= 2(-1) - 1 \cdot (-1)$

$= -1$

(f) $A^{-1}$

$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & -5 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\rightarrow \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$A^{-1}$
3. (20 pts)

(a) Find symmetric equations for the line that passes through a point \( x_0 = (1, 2, 3, 0) \) and is parallel to another line \( L : \mathbf{x} = (1 + s, -s, s/2, 3 - 2s) \).

The line in question thus has the following parametric equations

\[
\frac{\mathbf{x} - (1, 2, 3, 0)}{s} = \frac{\mathbf{1} - \mathbf{x}_0}{s} = \frac{\mathbf{1} - \mathbf{x}_0}{2}
\]

or equivalently

\[
x_1 = 1 + s, \quad x_2 = 2 - s, \quad x_3 = 3 + \frac{1}{2}s, \quad x_4 = -2s
\]

Solving for \( s \) and equating all,

\[
\frac{x_1 - 1}{1} = \frac{x_2 - 2}{-1} = \frac{x_3 - 3}{\frac{1}{2}} = \frac{x_4 - 0}{-2}
\]

(b) Find parametric equations for the plane \( P \) that contains the line \( L \) in part (a) and the origin.

\[
\mathbf{u} = (0, 0, 0, 0) - (1, 0, 0, 3) = (-1, 0, 0, -3)
\]

\[
\mathbf{x} = (1, 0, 0, 3) + s (1, -1, \frac{1}{2}, -2)
\]

\[
+ t (-1, 0, 0, -3)
\]

\[
= (1 + s - t, -s, \frac{1}{2}s, 3 - 2s - 3t)
\]
4. (20 pts) Given the following vectors in $\mathbb{R}^3$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$ 

(a) Determine if $b = \begin{pmatrix} 5 \\ -1 \\ 5 \end{pmatrix}$ lies in the set spanned by the four vectors above.

**Observation:** $\overrightarrow{b} = 5\overrightarrow{v}_3 - \overrightarrow{v}_4$ (or any other choice, e.g. $\overrightarrow{b} = 4\overrightarrow{v}_3 + \overrightarrow{v}_2$)

Thus $\overrightarrow{b}$ is in the spanning set $\mathbb{L}[\overrightarrow{v}_1,...,\overrightarrow{v}_4]$.

More formally,

$$\begin{pmatrix} 5 \\ -1 \\ 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 & 5 \\ 1 & -1 & 0 & 1 & -1 \\ 1 & 1 & 1 & 0 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & -2 & -1 & 1 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $A\overrightarrow{z} = \overrightarrow{b}$ has solutions,

$\overrightarrow{b}$ lies in the set spanned by $\overrightarrow{v}_1,...,\overrightarrow{v}_4$.

(b) Is $\mathbb{L}[v_1,v_2,v_3,v_4] = \mathbb{R}^3$ true? Please justify your answer.

**False**, because $\text{rank} A = 2 \neq 3$ (dimension of $\mathbb{R}^3$).
5. (15 pts) Find a sequence of elementary matrices $E_1, \ldots, E_k$ such that

$$E_k \cdots E_1 \cdot \begin{pmatrix} 1 & 3 & 0 & 0 \\ -1 & -3 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix} = \text{RREF}.$$ 

Also, specify the RREF of the given matrix.

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ -1 & -3 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix} \quad E_1 = E_{12}(1)$$

$$\rightarrow \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad E_2 = E_{23}$$

$$\rightarrow \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad E_3 = E_{2}(\frac{1}{2})$$

$$\rightarrow \text{RREF}$$

$$E_{2}(\frac{1}{2}) \cdot E_{23} \cdot E_{12}(1) \cdot \begin{pmatrix} 1 & 3 & 0 & 0 \\ -1 & -3 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix} = \text{RREF}.$$ 

Note the order!
6. (15 pts) Compute the determinant of the following matrix (hint: use definition)

\[
A = \begin{pmatrix}
2 & 0 & 0 & 0 \\
7 & 3 & 4 & 8 \\
-9 & 2 & 3 & 12 \\
3 & 0 & 0 & 1
\end{pmatrix}
\]

Can the columns of \( A \) be a basis for \( \mathbb{R}^4 \)?

\[
\det A = 2 \begin{vmatrix} 3 & 4 & 8 \\
2 & 3 & 12 \\
0 & 0 & 1 \\
\end{vmatrix} = 2 + 0 + 0 + 0 \quad \text{(expand along 1st row)}
\]

(expand along last row) = 2 \cdot \left( (-1)^{3+3} \begin{vmatrix} 3 & 4 \\
2 & 3 \\
\end{vmatrix} + 0 + 0 \right)

= 2 (9 - 8)

= 2

Since \( \det A \neq 0 \), we have \( \text{rank} \, A = 4 \), i.e. nonsingular, the columns of \( A \) form a basis for \( \mathbb{R}^4 \).
7. (10 pts) Let $A$ be an $n \times n$ matrix. All real numbers $\lambda$ satisfying $\det(A - \lambda I) = 0$ are called the eigenvalues of the matrix $A$. Use this definition to find all the eigenvalues of the following matrix

$$
\begin{pmatrix}
2 & 1 \\
2 & 3
\end{pmatrix}
$$

\[ 
\det \left( A - \lambda I \right)
\]
\[ 
= \det \left( \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right)
\]
\[ 
= \det \begin{pmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{pmatrix}
\]
\[ 
= (2-\lambda)(3-\lambda) - 2
\]
\[ 
= \lambda^2 - 5\lambda + 4 = 0
\]
\[ 
(\lambda-1)(\lambda-4) = 0
\]
eigenvalues: \( \lambda = 1, \lambda = 4 \)