FUNCTIONS

A mathematical function is an idealized “black box” with an input port and an output port. Inputs and outputs are typically scalars, multi-scalars, or vectors (what distinguishes vectors from multi-scalars is that vectors can be added and multiplied by scalars, according to well known laws; see axioms of a vector space). The domain of a function is the set of all its admissible inputs. Each input produces a single output. It is permissible for different inputs to produce the same output; for example the function $y = x^2$ produces the same output (value of $y$) for the inputs $x$ and $-x$. The outputs are assigned within a set, called the target set. The subset of the target set that contains all the outputs is the range of the function. A function is determined by specifying its domain, its target set and the rule that assigns the output to an input. Specifying the domain and the target sets is often omitted when obvious, a practice that sometimes leads to confusion. Functions are represented abstractly, using notations such as $y = f(x)$. In this notation, $x$ represents an input, $y$ represents the corresponding output and $f$ represents the rule that assigns the output to an input. When the inputs of a function are vectors, the function is often referred to as a field (not to be confused with algebraic fields). We talk of a scalar field when the output is a scalar, or of a vector field when the output is a vector. Thus, the function, that
specifies the ambient temperature at each point of the atmosphere at some fixed time, is a scalar field; the function, that gives the earth’s magnetic field in space, is a vector field.

Some functions are referred to as operators. Common wordings are that the operator operates on, or acts on, or is applied to the input, to produce the output. An example of this is a matrix, that acts on a vector through matrix multiplication to produce another vector, in symbols,

$$A x = y.$$  \hspace{1cm} (1)

The term “operator” is also used for functions in which the inputs and outputs are themselves functions, as occurs in the operation of taking the derivative. Consider, for example, the operator with domain all differentiable real valued functions of a real variable $x$ and with target space all real valued functions of $x$. We denote the operator as $\frac{d}{dx}$ and the action of the operator as

$$\frac{d}{dx} f = f'.$$  \hspace{1cm} (2)

We may consider more general differential operators, $\frac{d}{dx} + q(x)$ (of first order), $\frac{d^2}{dx^2} + p(x) \frac{d}{dx} + q(x)$ (of second order), etc. They are named ordinary differential operators to distinguish them from partial differential operators of multivariate and vector calculus, that take the derivative with respect to two or more variables. Examples of such partial differential operators of are:

- the translation operator in one space dimension $\frac{\partial}{\partial t} + a \frac{\partial}{\partial x}$, where $a$ is the speed of translation.

- the translation operator in two space dimensions $\frac{\partial}{\partial t} + a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y}$, where $a_1$ and $a_2$ are the speeds of translation in the $x$ and $y$ directions respectively. Translation in three space dimensions is similar.

- the Laplacian operator $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ in two variables or $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ in three variables. The Laplacian operator in one variable is simply $\Delta = \frac{\partial^2}{\partial x^2}$, an ordinary
differential operator.

- the **heat operator** in one, two or three space variables, respectively, \( \frac{\partial}{\partial t} - a^2 \frac{\partial^2}{\partial x^2} \), \( \frac{\partial}{\partial t} - a^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \) and \( \frac{\partial}{\partial t} - a^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \). The heat operator is written more compactly as \( \frac{\partial}{\partial t} - a^2 \Delta \). The constant \( a^2 \) is called the **diffusion constant**.

- the **wave operator** in one two or three space variables \( \frac{\partial^2}{\partial t^2} - a^2 \Delta \)

An operator \( \mathbb{L} \) is **linear**, if it satisfies the **linearity relation**

\[
\mathbb{L}(c_1 f_1 + c_2 f_2) = c_1 \mathbb{L} f_1 + c_2 \mathbb{L} f_2,
\]

for all inputs \( f_1 \) and \( f_2 \) and all scalars \( c_1 \) and \( c_2 \). In three different but equivalent wordings this says:

- The output of a linear superposition of two inputs equals the (same) linear superposition taken over the outputs of the two individual inputs.

- Taking a linear superposition of two inputs and then applying the operator produces the same result as applying the operator to the individual inputs and then taking the linear superposition of the two outputs.

- The operations of applying the operator and taking a linear superposition commute.

**Remark:** All the differential operators considered above as examples are linear.

The operation of **linear superposition**, as used here, constitutes a broadening in two respects of the operation of **linear combination** in linear algebra. Superposition allows (and we will use eventually), the following.

1. **Linear superposition of infinitely many functions,**

\[
\sum_{i=1}^{\infty} c_i f_i.
\]
Recall that the sum in a linear combination has finitely many terms.

2. **Linear superposition with respect to a continuous parameter**, 

\[
\int_{p_0}^{p_1} c(p) f(x, p) dp.
\]  

(5)

Superposition now occurs through integration with respect to the continuous parameter \(p\), contrary to a linear combination of linear algebra, where superposition occurs through addition over a discrete variable, the index \(i\). The parameter \(p\) is the continuous analogue of the discrete parameter \(i\).

We now consider an example of an operator that comes from heat transfer. A rod of length \(L\), considered for simplicity as one-dimensional, is insulated all around, except at the endpoints \(x = 0\) and \(x = L\). At these two points, the temperature \(T\) is fixed to \(T = 0\). Given the initial temperature distribution \(T_i(x)\) along the rod \((0 < x < L)\), find the temperature distribution \(T(x, t)\) along the rod at times \(t > 0\). With the aid of mathematical equations that model natural laws (they boil down to equation (6) below), the physical question takes the form of the mathematical problem of finding the function \(T(x, t)\), that satisfies the partial differential equation

\[
\frac{\partial T}{\partial t} - a^2 \Delta T = 0, \quad 0 < x < L, \quad t > 0,
\]  

(6)

the boundary conditions

\[
T(0, t) = 0, \quad T(L, t) = 0,
\]  

(7)

and the initial condition

\[
T(x, 0) = T_i(x).
\]  

(8)

Equation (6) is known as the heat equation, or the diffusion equation. The problem belongs to a class of initial value problems (IVP) or evolution problems. It is also
often characterized as an **initial-boundary value problem** to cite the fact that it involves both types of conditions. In solving the mathematical problem, we are able to **construct** a function $G(x, t, s)$ such that the answer to the question is given by the **solution formula**

$$T(x, t) = \int_0^L G(x, t, s)T_i(s)ds,$$ \hspace{1cm} (9)

for *any* choice of the initial distribution $T_i(x)$ within a wide family of distributions. The function $G(x, t, s)$ and the integral can be interpreted as follows. A unit value of the *initial* temperature over the infinitesimal interval $(s, s + ds)$ on the rod contributes $G(x, t, s)ds$ units of temperature at the point $x$, at time $t$. This contribution from $s$ to $x$ is proportionate to the amount of units of initial temperature at $(s, s + ds)$. The temperature at point $x$ on the rod at time $t$ is obtained through the process of integration, which is the continuum limit of adding all such contributions that arise from partitioning the rod into infinitesimal intervals.

The variable $t$ is a **parameter** in the solution formula; as $t$ increases, the formula describes the **evolution** of the temperature distribution in the rod.

Let us now keep the value of time $t$ fixed at some particular positive value, that we regard as the final time. Let $T_f(x)$ be the temperature distribution at the final time. The solution formula describes the action on an initial heat distribution (input) to produce the corresponding final distribution (output). Symbolically, the integral formula (9) describes an operator $\mathcal{G}$, that acts on $T_i$ to produce $T_f$,

$$T_f = \mathcal{G}T_i.$$ \hspace{1cm} (10)

Formula (9) implies (prove this) that for any scalars $c_1$ and $c_2$,

$$\mathcal{G}(c_1T_1 + c_2T_2) = c_1\mathcal{G}T_1 + c_2\mathcal{G}T_2.$$ \hspace{1cm} (11)

Thus, the solution operator $\mathcal{G}$ is linear.
LINEAR ANALYSIS

Linearity is important in the study of differential equations. Some of the ODE and all the PDE that we will study are linear, taking the forms,

\[
Lu = 0 \text{ (linear homogeneous, [H])}, \quad Lu = f \text{ (linear nonhomogeneous, [NH])}, \quad (12)
\]

where \( L \) is a differential operator. Let us assume that \( u_1 \) and \( u_2 \) are solutions of the linear homogeneous equation (12). This means that \( Lu_1 = 0 \) and \( Lu_2 = 0 \). We see that a linear combination of these two, \( u = c_1u_1 + c_2u_2 \), is also a solution of the equation \( Lu = 0 \), since, by the linearity property,

\[
L(c_1u_1 + c_2u_2) = c_1Lu_1 + c_2Lu_2 = 0 + 0 = 0. \quad (13)
\]

Thus, we learn that

- \textit{Any linear combination of solutions of a linear homogeneous equation is also a solution of the equation.}

This obvious statement is known as the \textbf{principle of linear superposition} of solutions and is the foundation for obtaining the \textbf{general solution} (that is all solutions) of a linear homogeneous equation. The statement can be rephrased as follows.

- \textit{The set of all solution of a linear homogeneous equation is a vector space (see below)}

To obtain the solution of a linear homogeneous equation, it suffices to find a \textbf{basis} of this vector space. Every solution is then represented as a linear combination of these basis solutions. When dealing with PDE, we will have to extend the linear algebra notion of a basis, to include bases with infinitely many vectors.

For the general solution of \textit{any} linear nonhomogeneous equation, we have the simple but powerful statement (can you prove it?)
To obtain the solution of a linear nonhomogeneous equation, when we already know the general solution of the corresponding homogeneous equation, it is sufficient to find one solution of the nonhomogeneous equation, say, \( u_{\text{particular}} \). Then

\[
\text{General solution of } [NH] = \text{General solution of } [H] + u_{\text{particular}}.
\]

As indicated in the subscript, the one solution of the nonhomogeneous equation that we need to obtain, is referred to as a particular solution of the nonhomogeneous equation.

A vector space is a set of elements called vectors. Vectors can be added (vector addition) and multiplied by a scalar (scalar multiplication). In both operations, the result is a vector. In our course, the scalars will be either real (real vector space), or complex (complex vector space). The two operations satisfy the following axioms.

1. Addition is associative \(((x_1 + x_2) + x_3 = x_1 + (x_2 + x_3))\), commutative \((x_1 + x_2 = x_2 + x_1)\), there is a zero vector \((x + 0 = x)\) and each vector \(x\) has a unique opposite vector \(\tilde{x}\), defined by \(x + \tilde{x} = 0\).

2. Scalar multiplication is associative \(((k_1k_2)x = k_1(k_2x))\); it is distributive in both \(k\) and \(x\), that is, \(k(x_1 + x_2) = kx_1 + kx_2\) and \((k_1 + k_2)x = k_1x + k_2x\), and finally, \(1x = x\).

Important Remark: Functions can be added and multiplied by scalars in accordance to these properties, so functions are vectors.

It is assumed as a prerequisite that the student has thorough understanding of basic matrix operations and the linear algebra concepts of linear combination of a set of vectors and what it means for the linear combination to be trivial, linear span of a set of vectors, linear independence of a set of vectors, basis, subspace of a vector space, nullspace and range of a matrix or linear operator, rank of a matrix, inverse of a square matrix, eigenvalues and eigenvectors of a square matrix and their calculation.

The eigenvalue problem for a linear operator \(L\) is to find scalars \(\lambda\) and corresponding vectors (functions) \(\psi\), with \(\psi\) not identically equal to zero, that satisfy the eigenvalue
Excluding $\psi \equiv 0$ makes sense. If $\psi \equiv 0$, any scalar $\lambda$ would satisfy the equation (15).

The following equivalent statements use the above terminology. They are equivalent in the sense that each of them is true if and only if any other one of them is true. The equivalence of the statements either reviews aspects of linear algebra, or rephrases earlier statements made in this overview. In the statements, $A$ is a square matrix, $x$ and $b$ are vectors.

- There exists a vector $x \neq 0$ that satisfies the equation $Ax = 0$.

- The matrix $A$ has a nontrivial nullspace.

- The matrix does not have a full range.

- The matrix $A$ does not have full rank.

- The equation $Ax = 0$ has infinitely many solutions $x$ for some values of the vector $b$ (including the value $b = 0$). It has no solutions for other values of $b$.

- The matrix $A$ has a zero eigenvalue.

- The determinant of the matrix $A$ equals zero.

- If $x_p$ is a particular solution of the equation $Ax = b$, where $b \neq 0$, then every solution of this equation can be written as $x = x_p + x_0$, where $x_0$ is an element of the nullspace of $A$.

Good study question: Create a similar set of equivalences if the matrix $A$ is not square, by rephrasing some of the statements and discarding others as non-applicable.

Remark: If $L$ is a linear operator and $u_p$ is a particular solution of the equation $Lu = f$, where $f$ is not identically zero, then every solution of this equation can be written as $u = u_p + u_0$, where $u_0$ is an element of the nullspace of $L$. 

\[ \mathbb{L} \psi = \lambda \psi, \quad \psi \neq 0. \]
ORDINARY DIFFERENTIAL EQUATIONS (ODE) OF FIRST ORDER

An ordinary differential equation of the first order involves a function \( y(x) \) and has the general form,

\[
\frac{dy}{dx} = f(x, y). \tag{16}
\]

At each \( x \), the ODE provides the value of the first derivative (slope) \( \frac{dy}{dx} \), in terms of two pieces of information, the value of \( x \) itself and the value of \( y \) at \( x \). The information is in the function \( f(x, y) \), which is often referred to as a slope field. The challenge is to find all functions \( y(x) \) that satisfy the equation. Graphically, this means drawing all curves in the \((x, y)\) plane whose slope at each point \((x, y)\) equals \( f(x, y) \)(see figures in text). To limit the range of solutions, we, additionally, consider the initial condition

\[
y(a) = b \tag{17}
\]

where \( a \) and \( b \) are given numbers. The system of equations (16) and (17) constitutes an initial value problem.

Starting at \( x = a \), we know the value of \( y \) (it equals \( b \)) and we also know the slope \( y' = f(a, b) \). We might be tempted to start drawing the graph of \( y(x) \) on the \((x, y)\) plane using these data, however, as we leave point \((a, b)\), the value of \( f \) and hence of the slope changes and will need to be updated continuously. We are led to try (do it on a piece of paper or better with the aid of a computer) the following scheme (Euler scheme) for obtaining an approximate solution. We choose a small step \( \Delta x \) in the \( x \) variable. We vary \( x \) from the value \( a \) to the value \( a + \Delta x \), holding the value of the slope fixed at \( f(a, b) \). At constant slope, the graph of the approximate solution in this interval is a (straight) line from point \((a, b)\) to point \((a + \Delta x, b + \Delta y_1)\), where \( \Delta y_1 = b + f(a, b) \Delta x \). At \( x = a + \Delta x \), we update the slope to the value \( f(a + \Delta x, b + \Delta y_1) \) and hold this value of the slope in the interval from \( x = a + \Delta x \) to \( x = a + 2\Delta x \), where we re-update the slope and so on. We repeat the scheme with the steps \( \Delta x \) taken in the direction of decreasing \( x \).
Advanced analysis shows that if \( f(x, y) \) is continuous in \( x \) and has continuous derivatives in \( y \) everywhere in the \((x,y)\) plane, then,

1. as \( \Delta x \) approaches zero, the scheme converges to a curve \( y(x) \).

2. the curve proceeds on either side (forward or backward from \( x = a \)), with \( x \) or \( y \) or both tending to positive or negative infinity.

3. \( y(x) \) is an exact solution of the ODE (16) and satisfies the initial condition (17) (existence of the solution of the IVP).

4. \( y(x) \) is the only solution of the IVP (uniqueness of the solution of the IVP).

5. If the hypotheses are only known to be satisfied in a neighborhood of point \((a,b)\), the existence and uniqueness of the solution are guaranteed only in a (possibly smaller) neighborhood of this point.

Imagine now repeating this construction taking the initial point \((a,b)\) to be a different point of the already constructed solution graph. By the uniqueness of the solution, the new and the old graphs will coincide (elaborate). A new graph will be obtained, if the initial point is chosen off the first graph. The two graphs will have no point in common (explain). Finally, by performing this construction at all points, one may imagine the \((x,y)\) plane filled (or foliated) by such graphs. Consider this foliation in the following trivial examples of ODE, in which an explicit formula for the solution \( y(x) \) can be easily written down.

- \( f(x,y) = 0; \quad y' = 0, \quad y(x) = \text{constant} \).
  
The solution graphs are horizontal lines that fill the \((x,y)\) plane.

- \( f(x,y) \) is independent of \( y \), call it \( f(x) \); \( y' = f(x), \quad y(x) = \int_0^x f(s)ds + \text{constant} \).
  
The solution graphs are curves that fill the \((x,y)\) plane. They are obtained by translating one of them parallel to the \( y \) axis, varying the value of the constant.
\[ f(x, y) = y : \quad y' = y, \quad y(x) = ce^x, \text{ where } c \text{ is a constant.} \] By varying the value of the constant, one fills the \((x, y)\) plane with the graphs.

In general, a formula for the general solution of an ODE contains a free constant, known as the **constant of integration**. If one has a formula for the general solution and wants a specific solution of the ODE that satisfies the initial condition (17), one inserts \(x = a\) and \(y = b\) into the general solution. This leaving the constant as the only unknown, to be determined by solving the equation. For example, in the second of the above examples with \(f(x)\) assumed to be known and with initial condition \(y(1) = 3\), the equation that determines the constant \(c\) is \(3 = \int_0^1 f(s)ds + c\).

There are several types of first order ODE, for which a formula for the general solution can be obtained in terms of integrals. We call such ODE **integrable** or **solvable**, even when the integrals cannot be calculated by a formula.

We list the following types of differential equations that are integrable.

1. **Separable ODE:** \[ \frac{dy}{dx} = f(x)g(y) \]
   - Write in form that separates variables \(x\) and \(y\): \[ \frac{dy}{g(y)} = f(x)dx \]
   - Take the indefinite integral on both sides,
     \[ \int_{G(y)} \frac{dy}{g(y)} = \int_{F(x)} f(x)dx + c \]
   - General solution: \(G(y) - F(x) = c\). The solution is **implicit** (not in the forms \(y = h(x)\) or \(x = k(y)\)).

2. **Exact ODE:** \( h_x dx + h_y dy = 0 \), or (review calculus complete differentials) \( dh = 0 \)
   - Notation: \( h = h(x, y)\), \( h_x = \frac{\partial h}{\partial x}\), \( h_y = \frac{\partial h}{\partial y}\)
   - General solution: \( h(x, y) = c\).
   - The ODE \( M(x, y)dx + N(x, y)dy = 0 \) is exact, if \( M_y = N_x \).
– Condition \( M_y = N_x \) allows finding a function \( h(x,y) \), that satisfies \( h_x = M \) and \( h_y = N \). Notice that then, the condition \( M_y = N_x \) is simply the \textit{compatibility condition}, \( h_{xy} = h_{yx} \).

– Solution: \( h(x,y) = \int M(x,y)\,dx + k(y) \) (convenient choice of indefinite integral): Find \( k(y) \) from \( h_y = N(x,y) \)

• It may be possible to make a nonexact ODE exact, by multiplying the ODE by an \textit{integrating factor}. (example in next bullet).

3. \textbf{Linear ODE}: \( \frac{dy}{dx} + p(x)y = f(x) \).

• The general solution to the homogeneous ODE \( (f(x) \equiv 0) \) is

\[
y_{[H]}(x) = ce^{-\int p(x)\,dx}
\]  

(18)

Use a convenient constant of integration when calculating the integral.

• In order to find a particular solution, let \( W(x) = e^{-\int p(x)\,dx} \), be a solution of the homogeneous ODE.

• Verify that a particular solution is given by the formula

\[
y_{\text{particular}}(x) = \int_{x_0}^{x} \frac{W(x)}{W(s)} f(s)\,ds
\]  

(19)

where \( x_0 \) is any convenient lower limit of integration. Notice that the \( W(x) \) can be taken out of the integral, the expression though looks more symmetric and mnemonically easier as it is.

• The general solution of the full ODE is

\[
y(x) = ce^{-\int p(x)\,dx} + \int_{x_0}^{x} \frac{W(x)}{W(s)} f(s)\,ds
\]  

(20)

• The algebra for deriving the integral formula goes like this. Multiply the ODE
by the integrating factor $J = 1/W$, to obtain $Jy' + pJy = Jf$. Insert $pJ = J'$ to obtain $Jy' + J'y = Jf$. Apply the product rule: $(Jy)' = Jf$ and integrate from $x_0$ to $x$. Finally, solve for $y$. Notice that all this is in agreement with the earlier description of the solution of linear nonhomogeneous equations.

4. **Homogeneous ODE:** (do not confuse with “linear homogeneous”): $\frac{dy}{dx} = f(x,y)$, where $f(x,ux) = g(u)$ ($x$ drops out)

   - Transforms to a separable ODE $u' = G(u)$, through the substitution: $y = ux$, $y' = xu' + u$, where $u$ is a new dependent variable ($u' = du/dx$).

5. **Method of substitution:** Sometimes, a clever substitution (change of variable) transforms an ODE to an integrable one.

   *All general solutions of first order ODE contain a constant of integration*

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**The notion of phase space:**

When the slope field of an ODE does depend on $x$, that is when $\partial f/\partial x$ is not identically equal to zero, we refer to the $(x,y)$ plane as the **phase-plane** of the ODE. The idea behind this, is that the ODE describes the evolution of a system, whose possible **states** are represented by points of the plane. The evolving state is pictured as a particle moving on the $(x,y)$ plane in an imagined time $t$. The coordinate $x$ is thought of as evolving according to $x = t$ (in essence $x$ is time), while $y$ evolves in this time as dictated by the function $f(x,y)$ in the ODE. Systems in diverse fields, like physics, finance, ecology, ... can be represented mathematically in this way.

When the slope field $f(x,t)$ is independent of $x$, that is when the ODE has form $dy/dx = f(y)$, the time variable $x$ does not affect the evolution of $y$ and the ODE is called **autonomous**. The state of the system is represented only by $y$ and the $y$ axis is the **phase-line**. The zeros of the function $f$, that is the values of $y$ for which $f(y) = 0$, are called
equilibrium points of the ODE. If \( y_0 \) is an equilibrium point, then the constant function \( y(x) = y_0 \) is a solution, that is referred to as an equilibrium solution. An equilibrium point (solution) is characterized as stable or unstable, according to whether the solution with initial condition \( y_1 \) that is close to \( y_0 \) remains close to \( y_0 \) as time becomes ever larger, or it does not. We define the notions more precisely when we discuss ODE systems.

Example of an autonomous ODE:

\[
y' = y(y - 1)(y - 2)
\]  

(21)

The student should be able to pick out immediately the three equilibrium points \( y = 0, 1 \) and 2 and characterize each as stable or unstable, through simple observations such as:

1. The direction of motion and the instantaneous speed of a particle, located at some point on the \( y \) axis (phase-line) are obtained directly from the ODE.

2. A particle located at an equilibrium point stays there forever, forward and backward in time (hint: use the uniqueness of solutions).

3. A particle that is not on an equilibrium point is in motion (does not have zero speed).

4. A particle headed towards an equilibrium point on the phase-line, does not reach it in finite time. Check this, in the example \( y' = y \), by solving the ODE.

5. A particle headed towards \( \pm \)infinity may possibly reach it in finite time. Check this, in the example \( y' = y^2 \), by solving the ODE. Give an example in which it takes the particle infinite time to reach infinity.

An orbit of an autonomous first order ODE is a maximal connected path of a particle along the \( y \) axis. It can be an equilibrium point, or an open interval connecting two equilibrium points, or an open half-axis from an equilibrium point to \( \pm \)infinity, or the whole line (absence of equilibrium points, as in \( y' = y^2 + 1 \)).
Examples of Derivation of and Modelling with ODE

1. Bank account with continuous interest compounding.

   \( y(t) \): Account balance at time \( t \) (in dollars)

   \( r(t) \): Rate of interest at time \( t \) (in dollars of interest per dollar in account, per unit time)

   \( g(t) \): Rate of deposit minus rate of withdrawal (in dollars per unit time)

   ODE: \( \frac{dy}{dt} = r(t)y + g(t) \).

   Verify that each of the three terms of the equation are in dollars per unit time.

2. Population dynamics (Verhulst model)

   \( y(t) \): Population at time \( t \).

   Model ODE: \( \frac{dy}{dt} = ry - \frac{r}{K}y^2 \), \( r \) and \( K \) are positive constants.

   The first term on the right models intrinsic growth of population, that is growth in the absence of limiting factors. The second term models limiting factors, like availability of space and of resources. Discuss what happens to the model population in the long run.

3. Chemical reaction: A mole of substance P reacts with a mole of substance Q to produce a mole of substance X, in reaction symbols, \( P + Q \rightarrow X \).

   \( x(t) \): Concentration of substance X at time \( t \)

   \( p \): Initial concentration of substance P.

   \( q \): Initial concentration of substance Q.

   Argue, that the ODE

   \[ \frac{dx}{dt} = \alpha(p - x)(q - x), \quad \alpha \text{ is a constant}, \]

   is a reasonable model for the reaction rate. Discuss what will happen in the long run.
Bifurcation


AUTONOMOUS FIRST ORDER ODE SYSTEMS

Autonomous first order ODE systems have form

$$\frac{dy}{dt} = f(y), \quad (22)$$

where \( y \) is a vector in \( \mathbb{R}^n \) and \( f(y) \) is a vector field. As earlier, the term autonomous reflects the fact that the evolution of \( y \) is not affected by the values of the time-variable \( t \).

Written in scalar form, the system, for example in \( \mathbb{R}^2 \), has

$$\begin{cases} \frac{dy_1}{dt} = f_1(y_1, y_2) \\ \frac{dy_2}{dt} = f_2(y_1, y_2). \end{cases} \quad (23)$$

The notion of orbit, introduced for one-dimensional autonomous ODE, as a maximal path of a particle in phase space persists. Orbits are now curves in phase-space that may be quite complicated. The notions of phase-space, equilibrium points or solutions and stable or unstable solutions apply to these ODE systems. The phase space of the ODE system (22) is \( \mathbb{R}^n \). Its equilibrium points are the points of \( \mathbb{R}^n \) that satisfy

$$f(y) = 0. \quad (24)$$

For example, a point \((y_1, y_2)\) is an equilibrium point of (23), if it satisfies the equations

$$\begin{cases} f_1(y_1, y_2) = 0 \\ f_2(y_1, y_2) = 0 \end{cases} \quad (25)$$
An equilibrium point \( y_0 \) is **stable** if an arbitrarily small sphere \( S_0 \) centered at \( y_0 \), traps all particles that start within some (smaller) sphere \( S_1 \), again centered at \( y_0 \), of radius that is allowed to depend on the radius of \( S_0 \). Otherwise, the equilibrium point is **unstable**. The equilibrium point is **asymptotically stable** if all solutions starting within some sphere centered at \( y_0 \), converge to \( y_0 \), in other words, if they satisfy \( y \to y_0 \) as \( t \to +\infty \).

From the prerequisite course of linear algebra, the student is familiar with the case in which the vector field \( f(y) \) depends linearly on \( y \), that is, it is given by a constant *square matrix* \( A \) acting on \( y \).

\[
\frac{dy}{dt} = Ay, \tag{26}
\]

We give a brief review of deriving the solution of (26).

1. **Guess**: There are solutions of the form \( y = v e^{\lambda t} \), where \( \lambda \) is a scalar and \( v \) is a *nonzero* constant vector (by constant is meant independent of \( t \)).

2. Insert the guess into equation (26) to obtain

\[
Av = \lambda v, \quad v \neq 0. \tag{27}
\]

*Finding all values of \( \lambda \) and the corresponding vectors \( v \) that satisfy both relations (27) constitutes the **eigenvalue problem for the matrix** \( A \). The fact that solving eigenvalue problems helps in finding solutions of differential equations, underlines the **significance of eigenvalue problems** in mathematics.*

3. In order to find the eigenvalues, one recalls \( \lambda v = \lambda I v \), where \( I \) is the identity matrix and uses the distributive law of matrix multiplication to write problem (27) as

\[
(A - \lambda I)v = 0, \quad v \neq 0, \tag{28}
\]
Since $\mathbf{v} \neq 0$, the matrix $A - \lambda I$ is not invertible (why?), thus,
\[
\det(A - \lambda I) = 0
\] (29)

The determinant is a polynomial of degree $n$, known as the **characteristic polynomial of the matrix** $A$. Thus it produces $n$ eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, some, possibly repeating.

4. Inserting each distinct eigenvalue in (28), we find the eigenvector(s) corresponding to the eigenvalue by solving the system. If a wrong value of the eigenvalue is inserted, the system produces only the zero solution (why?).

5. If the procedure yields $n$ linearly independent eigenvectors, $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$, (this means a basis of $\mathbb{R}^n$), the **general solution** to the ODE system (26) is
\[
y(t) = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \mathbf{v}_2 e^{\lambda_2 t} + \cdots + c_n \mathbf{v}_n e^{\lambda_n t},
\] (30)

where $c_1, c_2, \cdots, c_n$ are arbitrary constants.

6. If the procedure produces only $k$ independent eigenvectors, with $k < n$, more work is needed in order to obtain the $n - k$ missing solutions. Importantly, at least one entry of these missing vector solutions is a constant times $t^m e^{\lambda t}$, where $m = 1, 2, 3, \cdots$.

**LINEAR SECOND ORDER ODE**

The general form of a **linear second order ODE** for a function $y(x)$ is
\[
\frac{d^2 y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = f(x), \quad \text{or} \quad y'' + p(x)y' + q(x)y = f(x).
\] (31)

The ODE is **homogeneous** (labeled [H]) if $f$ is identically equal to zero ($f(x) \equiv 0$),
\[
y'' + p(x)y' + q(x)y = 0.
\] (32)
It is nonhomogeneous (labeled [NH]) otherwise.

**Remark:** We remind the reader for later use, that:

1. The problem of finding the general solution $y_{[H]}(x)$ of (32) can be rephrased as **finding the nullspace of the differential operator** $L = \frac{d^2}{dx^2} + p(x) \frac{d}{dx} + q(x)$.

2. According to the **principle of linear superposition**, the problem of finding the general solution of (31) amounts to finding the right side of the formula

$$y_{[NH]}(x) = y_{\text{particular}}(x) + y_{[H]}(x),$$

(33)

where $y_{\text{particular}}(x)$ denotes a **particular solution** of (31).

If the functions $p$, $q$ and $f$ are constant (independent of $x$), the ODE can be written as an **autonomous** first order system of ODE in two dimensions, by setting $y = y_1$ and $y' = y_2$. Equation (31) becomes (check)

\[
\begin{aligned}
    y'_1 &= y_2 \\
    y'_2 &= -py_2 - qy_1 + f.
\end{aligned}
\]

(34)

The $(y_1, y_2)$ plane is the **phase-plane** of this ODE system and hence of ODE (31). If there is dependence of any of $p$, $q$ and $f$ on $x$, then its representation as an ODE system requires introducing the new time variable $t$ (which we take equal to $x$). Denoting the derivative with respect to $t$ with a dot instead of a prime, we write equation (31) as the autonomous system

\[
\begin{aligned}
    \dot{x} &= 1 \\
    \dot{y}_1 &= y_2 \\
    \dot{y}_2 &= -p(x)y_2 - q(x)y_1 + f(x).
\end{aligned}
\]

(35)

The phase-space of this ODE system is three-dimensional.
Complex exponentials and complex trigonometric and hyperbolic functions

In the solution of second order linear ODE, one encounters exponentials $e^z$, where $z$ is a complex number. They are easy to deal with. Recall that the exponential $e^z$ is an entire function of $z$. A function $f(z)$ is entire, if it has a power series expansion with complex (includes real) coefficients,

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots,$$

and the series has infinite radius of convergence. In other words, the series converges for every complex value of $z$. The series for the exponential is

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots,$$

Using power series expansions, one verifies basic properties of the exponential. First and foremost, $e^{z_1 + z_2} = e^{z_1} e^{z_2}$ applies. Furthermore, if $b$ is a real number and $i$ is the imaginary unit ($i^2 = -1$), the exponential $e^{ib}$ is oscillatory,

$$e^{ib} = \cos b + i \sin b.$$

Putting these properties together, we have, for real $a$ and $b$,

$$e^{a+ib} = e^a (\cos b + i \sin b), \quad e^{a-ib} = e^a (\cos b - i \sin b)$$

We notice that the imaginary part of the exponent determines the phase of the oscillation, while the real part of the exponent determines the amplitude $e^a$.

The power series expansions make the connection between trigonometric and hyperbolic
functions,
\[ \cos(ib) = \cosh b, \quad \sin(ib) = i \sinh b \] (40)

**Constant Coefficients: the homogeneous case**

\[ y'' + py' + qy = 0, \quad p \text{ and } q \text{ are constants,} \] (41)

One may transform this to a first order ODE system and solve as described earlier. However, working directly for the second order scalar ODE is simpler.

1. **Guess:** \( y = e^{\lambda x} \),

2. Insert the guess into (41). You obtain the **characteristic equation** (it is a quadratic equation)
\[ \lambda^2 + p\lambda + q = 0. \] (42)

3. Let \( \lambda_1 \) and \( \lambda_2 \) be the two roots. Then, each of the functions \( e^{\lambda_1 x} \) and \( e^{\lambda_2 x} \) satisfies (is solutions of) ODE (41).

4. There are two cases,
   - **Case 1:** \( \lambda_1 \neq \lambda_2 \): The **principle of the linear superposition of solutions** dictates that
     \[ y(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}. \] (43)
     is a solution of (41) for any value of the constants \( c_1 \) and \( c_2 \). Further argument (based on the uniqueness of the initial value problem) demonstrates that it is, indeed, the **general solution** of (41).
   - **Case 2:** If \( \lambda_1 = \lambda_2 \) (refer to the double root as \( \lambda \)), one verifies by direct calculation (do it) that \( xe^{\lambda x} \) is also a solution. Again, by the **principle of linear superposition**
     \[ y(x) = c_1 e^{\lambda x} + c_2 xe^{\lambda x}. \] (44)
is a solution for any value of the constants $c_1$ and $c_2$, and (by the same argument as before) it is, indeed, the general solution of (41).

5. We conclude that the general solution of (41) is a two dimensional vector space. This space is clearly, the nullspace of the operator $\frac{d^2}{dx^2} + p \frac{d}{dx} + q$.

6. Assuming that the constants $p$ and $q$ are real (as we tacitly do throughout this course), the roots $\lambda_1$ and $\lambda_2$ are then either both real or they are complex conjugates of each other, that is $\lambda = r \pm i\omega$, where $r$ and $\omega$ are real. It is significant to observe that the real part $r$ of the root $\lambda = r + i\omega$ produces growth ($r > 0$) or decay ($r < 0$) in the solution. On the contrary, the imaginary part $\omega$ of the root produces oscillation with angular frequency $\omega$ (also called circular frequency). The frequency (reciprocal period) is $\omega/(2\pi)$. The classic physical example for this analysis is the mass-spring system.

**Mass-spring system:**

A point particle of mass $m$ moves along a line, subject to two forces,

1. a restoring force, proportional to the deviation of the particle’s position from a fixed central point on the line; the force may be thought as being produced by a linear spring,

2. a resisting force, proportional to the particle’s velocity, attributed to friction.

Let $y(t)$ be the position of the particle on the line and let $y = 0$ be the central point. The restoring force is then equal to $-ky(t)$, where the constant of proportionality $k$ is called the spring constant. The resisting force of friction acting on the particle equals $-\varepsilon \dot{y}(t)$, where the constant constant of proportionality $\varepsilon$ is the damping coefficient. By Newton’s second law (mass time acceleration equals force),

$$m\ddot{y} = -ky - \varepsilon \dot{y}, \quad \text{or} \quad \ddot{y} + py + qy = 0, \quad p = \frac{\varepsilon}{m}, \quad q = \frac{k}{m} \quad (45)$$
This is the ODE we are studying.

The roots of the characteristic equation are

$$\lambda = \frac{1}{2} \left(-p \pm \sqrt{p^2 - 4q}\right); \text{ when } p^2 - 4q < 0, \lambda = r \pm i\omega, \ r \in \mathbb{R}, \ \omega \in \mathbb{R}. \quad (46)$$

In the undamped system ($\varepsilon = 0, p = 0$) our ODE (45) is the **harmonic oscillator** (harmonic means “of a single frequency”).

$$\ddot{y} + \omega^2 y = 0, \ r = 0, \ \omega = \sqrt{q} = \sqrt{\frac{k}{m}}. \quad (47)$$

Using de Moivre’s formula, or by direct observation, we may write the general solution in terms of **trigonometric** functions,

$$y(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t), \quad (48)$$

justifying the name “harmonic oscillator”.

Increasing the amount of damping (increasing $\varepsilon$, and hence $p$ from the zero value) formula (46) indicates a drop in frequency $\omega$ and the appearance of a **negative** real part $r$ in the roots. The general solution is now.

$$y(t) = e^{rt}(c_1 \cos(\omega t) + c_2 \sin(\omega t)), \ r < 0. \quad (49)$$

There is still oscillation. However, as $t$ increases, the **amplitude** of the oscillation decays **exponentially**, due to factor with the real exponential. When damping reaches the critical value $\varepsilon = 2\sqrt{k/m}$, the radical in (46) is zero. We have the double real root $\lambda = r = -p/2$ and the general solution

$$y(t) = e^{rt}(c_1 + c_2 t), \ r < 0. \quad (50)$$
There is no more oscillation. At still higher damping values, the general solution involves two *negative roots* $r_1$ and $r_2$ and there is pure decay.

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \quad r_1 < 0, \ r_2 < 0. \tag{51}$$

The harmonic oscillator plays a central role in physics. It is derived in diverse physical and engineering situations, including electrical circuits and laser optics. A different way to say this is that the mass-spring system serves as a model for many physical systems. There are physical situations in which, instead of damping, there is *gain*; energy is not taken away from the system through *dissipation*, but supplied to the system externally. In this case, $p < 0$ (still $q > 0$). The real parts $r$, that were negative in our exposition above, are now positive. Instead of *exponential decay*, there is *exponential growth* of the solutions.

**Forced Harmonic Oscillator and Resonance**

Equation (45) is a balance of forces. Suppose that there is a time-dependent external force $f(t)$ acting on the mass particle, and that there is no damping or gain. The ODE describing the system is now nonhomogeneous,

$$\ddot{y} + \omega^2 y = f(t), \tag{52}$$

Since the general solution of the unforced ODE is known, all that is needed is a particular solution of (52). There are two approaches for finding particular solutions of the forced ODE if the general solution of the unforced (that is the homogeneous) ODE is known.

1. **Variation of parameters**: This provides the particular solution by an integral formula that works for every second order linear ODE (constant or variable coefficients) and for every forcing $f(t)$. We address this approach when we discuss variable coefficients.
2. Undetermined coefficients: This works mostly for constant coefficients and for certain classes of forcings. For these, the textbook contains a table that gives the right guess for the form of a solution. The guess depends on a number of coefficients that are determined by inserting the guess into the ODE.

Let us now take a closer look at the forced ODE (52). A physical system modeled by the ODE, “wants” to oscillate with frequency $\omega$. More precisely, this is what it does, if left alone ($f = 0$). We say that $\omega$ is its natural frequency. How will the system respond, if we force it with a different frequency? how will it respond, if we force it with the same frequency? Suppose

$$f(t) = \cos(\Omega t)$$

(53)

Remark: Including an amplitude $a$ in the forcing formula, as in $f = a \cos(\Omega t)$, does not add generality to our study. The ODE is linear. The effect will be to multiply the solution we obtain for (53) by the factor $a$.

Inserting the guess $y = A \cos(\Omega t)$, we obtain the particular solution

$$y(t) = \frac{\cos(\Omega t)}{\omega^2 - \Omega^2}$$

(54)

The forcing frequency prevails over the natural frequency, in this solution. One must remember though, that the general solution

$$y(t) = \frac{\cos(\Omega t)}{\omega^2 - \Omega^2} + c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

(55)

includes oscillations in the natural frequency as well.

We notice that the amplitude of the particular solution tends to infinity as the forcing frequency approaches the natural frequency, a phenomenon referred to as resonance. At exactly the resonant frequency $\Omega = \omega$, the particular solution breaks down, our guess was not good, we need another one. Inserting the guess $y = A \text{tint}$, we obtain the resonant
particular solution (recall, now $\Omega = \omega$)

$$y(t) = \frac{t}{2\omega} \sin(\omega t).$$  \hspace{1cm} (56)$$

The amplitude now grows without bound, linearly in time. For a physical, system, this means that at some point the modeling ODE will no longer apply. The system may break, like a wine-glass breaking when a violin is played continuously at its natural frequency; or like a suspension bridge (Tacoma Narrows) collapsing because gusts of wind could excite modes of oscillation at the bridge’s natural frequency of oscillation.

As the forcing frequency grows without bound, the amplitude of the particular solution tends to zero.

**Second order Linear ODE with Variable Coefficients**

We begin with the homogeneous ODE (32), which we repeat for convenience

$$y'' + p(x)y' + q(x)y = 0.$$  \hspace{1cm} (57)$$

1. All solutions of (57) form a two-dimensional vector space and can be represented as a linear superposition of two basis solutions $y_1(x)$ and $y_2(x)$. The pair of the basis solutions is referred to as a fundamental solution of the ODE.

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$  \hspace{1cm} (58)$$

For most ODE (57) with variable coefficients a solution cannot be calculated explicitly.

2. We say that two solutions $y_1(x)$ and $y_2(x)$ are linearly independent at $x$, if the vectors $(y_1, y_1')$ and $(y_2, y_2')$ are linearly independent at $x$. The Wronskian of any two solutions
$y_1(x)$ and $y_2(x)$ is the function of $x$ defined

$$W(y_1(x), y_2(x)) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x). \quad (59)$$

Thus, two solutions are are linearly independent at $x$, if and only if their Wronskian is not zero at $x$.

3. As a function of $x$, the Wronskian of two solutions of (57) satisfies the first order ODE below. (Exercise: Derive it). The ODE is solvable and provides the value of the Wronskian at some point $x$ in terms of its value at some other point $x_0$.

$$W' = -p(x)W, \quad W(x) = W(x_0)e^{-\int_{x_0}^x p(s)\,ds}. \quad (60)$$

4. The second relation (60) implies that any two solutions of (57), $y_1(x)$ and $y_2(x)$ are either linearly independent at all $x$ or are linearly dependent at all $x$. (how)?

5. **Reduction of order**: If one knows one solution of (57), say $y_1(x)$, one can obtain a formula for a second solution $y(x)$. Insert $y = cy_1$ into (57), where $c = c(x)$ (not a constant) is to be determined. Doing the product rule and algebra, one obtains a linear second order ODE for the function $c(x)$ that has form $a(x)c'' + b(x)c' = 0$. The fact that the term with $c$ is missing (has cancelled), makes this a solvable (first order separable) ODE for $c'$. With $c'(x)$ known, one obtains $c(x)$ through integration.

We now tackle the **nonhomogeneous** or **forced** equation

$$y'' + p(x)y' + q(x)y = f(x). \quad (61)$$
In analogy to the first order forced equation, we anticipate a general solution of the form

\[ y(x) = c_1y_1(x) + c_2y_2(x) + \int_{x_0}^{x} G(x, s)f(s)ds. \] (62)

where \((y_1, y_2)\) is a fundamental solution of the homogeneous ODE, \(x_0\) is a constant chosen arbitrarily and \(G(x, s)\) is a function to be determined. The expectation comes true. The function \(G\) is given by the formula

\[ G(x, s) = \frac{\begin{vmatrix} y_1(s) & y_2(s) \\ y_1(x) & y_2(x) \\ y_1(s) & y_2(s) \\ y_1'(s) & y_2'(s) \end{vmatrix}}{y_1(s)y_2'(s) - y_1'(s)y_2(s)}. \] (63)

In other words,

\[ y_{\text{particular}}(x) = \int_{x_0}^{x} \frac{y_1(s)y_2(x) - y_1(x)y_2(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)} f(s)ds. \] (64)

This is the variation of parameters formula. To verify its correctness, the reader should recall the differentiation formula,

\[ \frac{d}{dx} \int_{x_0}^{x} F(x, s)ds = F(x, x) + \int_{x_0}^{x} \frac{\partial F(x, s)}{\partial x}ds. \] (65)

**Series Solutions**

Sequences, series and convergence

A numerical sequence is an infinite array of (generally complex) numbers

\[ u_1, u_2, u_3, \ldots, u_n, \ldots \] (66)

We refer to the numbers as the terms of the sequence and to the subscripts as indices. Thus, in the sequence of the odd numbers 1, 3, 5, 7, \ldots, the term corresponding to the index
6 has value 11. A numerical sequence may be understood as a function, in which the inputs are the indices and the set of complex numbers is the target set. Thus, we refer to a “sequence $u_n$” in the same spirit as we refer to a “function $f(x)$”; indeed, sometimes we even write $u(n)$ instead of $u_n$ to make the point that we are dealing with a function, in particular one that has a discrete independent variable. To truncate a sequence at an index $k$ means to discard all terms with index beyond $k$. We often refer to the sequence of terms beyond a given index as the tail of the sequence. This terminology allows us to express the mathematical definition of the convergence of a sequence in plain language.

We say that a sequence $u_n$ converges to a number $l$, if the following statement holds.

- Given any tolerance $\varepsilon > 0$, a tail of the sequence exists, in which, every term approximates the number $l$ within the tolerance.

Loosely speaking, the terms of the sequence, eventually, come arbitrarily close to the value $l$. The number $l$ is called the limit of the sequence. As the tolerance $\varepsilon$ is made smaller, the cut-off index for the largest successful tail becomes larger.

An equivalent expression of the above statement, given in analytic terms is

- Given $\varepsilon > 0$, there is an index $N$ such that if $n > N$, then $|u_n - l| < \varepsilon$.

If we make the terms $u_n$ dependent on some continuous variable $x$, that is $u_n = u_n(x)$, we obtain a sequence of functions. The series may converge for some values of $x$ and not converge for others. For example, the sequence $u_n(x) = e^{nx}$, where $n = 1, 2, 3, \ldots$ converges when $x \leq 0$, but does not converge when $x > 0$.

In mathematics and in the natural sciences one is often faced with infinite sums of the form

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$  \hspace{1cm} (67)

We say that this formal series converges if the sequence $s_1, s_2, s_3, \cdots$ of partial sums

$$s_n = u_1 + u_2 + u_3 + \cdots + u_n$$  \hspace{1cm} (68)
converges. The formal series, then, acquires solid meaning,

\[ \sum_{n=1}^{\infty} u_n = S \]  

where \( S \) is the limit of the sequence of partial sums.

**Geometric series** are special series of particular practical and theoretical importance. Each term of a geometric series is given by the formula \( u_{n+1} = ru_n \), in which \( r \) is a fixed number, known as the **ratio** of the geometric series. Such formulae, which determine each term of a series from previous terms are called **recursive**. In this case, a term is obtained by multiplying the previous term by the ratio \( r \). The partial sum corresponding to index \( n \) is

\[ S_n = u_1(1 + r + r^2 + r^3 + \cdots) = u_1 \frac{1 - r^n}{1 - r} = \frac{u_1}{1 - r} - \frac{u_1 r^n}{1 - r}. \]  

We examine the last fraction.

\[ \frac{r^n}{1 - r} \begin{cases} \rightarrow 0, & \text{if } |r| < 1 \\ \text{diverges if } |r| > 1 \end{cases} \]  

The geometric series converges to

\[ \frac{u_1}{1 - r} \]  

if \( |r| < 1 \). It does not converge otherwise.

A comparison with the geometric series is at the basis of the proof of the well-known **ratio test** for the convergence of a series. Assuming that the sequence of the ratios \( |u_{n+1}/u_n| \) converges to some number \( \rho \), the test states that

\[ \begin{cases} \rho < 1, & \text{series (67) converges,} \\ \rho > 1, & \text{series (67) diverges,} \\ \rho = 1, & \text{the test is inconclusive.} \end{cases} \]
We are interested in series of the form

\[ u(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots. \quad (74) \]

They are called **power series**. Two power series can be formally added and multiplied in the obvious way.

We now use the ratio test to assess the convergence of power series for which the ratio \(|a_{n+1}/a_n|\) converges to some number \(m\). The absolute value of the ratio of two successive terms of the series (74), then converges to \(m|x|\) as follows.

\[ \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \left| \frac{a_{n+1}x}{a_n} \right| \to m|x|. \quad (75) \]

According to the ratio test, the series

\[
\begin{align*}
\text{series (74) converges when } & m|x| < 1 \\
\text{series (74) diverges when } & m|x| > 1 \\
\text{the test is inconclusive when } & m|x| = 1
\end{align*}
\]

(76)

Thus,

\[
\begin{align*}
m = 0, \quad & \text{series (74) converges for all } x \\
m > 0 & \text{ and if } \begin{align*}
|x| < \frac{1}{m}, \quad & \text{series (74) converges} \\
|x| > \frac{1}{m}, \quad & \text{series (74) diverges} \\
|x| = \frac{1}{m}, \quad & \text{the test is inconclusive}
\end{align*}
\end{align*}
\]

(77)

Thus, series (74) converges for all \(x\) in the complex plane at a distance less than \(1/m\) from the origin. It converges for all \(x\) if \(m = 0\). We say that

\[ R = \frac{1}{m} \]

(78)
is the **radius of convergence** of the series. Notice that, in a power series, \( x \) is allowed to take complex values.

Power series of the form of (74) have the following important properties (proofs omitted).

1. A power series either (a) converges in an open disc of the complex plane and diverges outside the disc (radius of convergence \( R > 0 \)), or (b) it converges for all complex values of \( x \) (radius of convergence \( R = \infty \)), or it diverges for all \( x \neq 0 \) (radius of convergence \( R = 0 \)).

2. In the interior of its disc of convergence (assuming \( R \neq 0 \)), a power series defines a function. A function, defined in this way, is referred to as an **analytic function**. If this function is the zero function, the power series has all its coefficients equal to zero.

3. A power series differentiated term by term yields a series that has the same radius of convergence as the original one (verify this in the case of the assumptions of the ratio test holding).

4. The differentiated power series represents the derivative of the function represented by the original series. Thus, an analytic function is **differentiable to all orders**.

**Remark:** Generally, a power series is centered at some point of interest \( x_0 \) that may be different from the origin. The series is then in powers of the deviation \( x - x_0 \). It is often convenient to redefine our \( x \) as being the deviation from \( x_0 \) and work with powers of \( x \).

**Example 1: Using PS to solve a first order ODE**

The ODE we solve is easy: \( y' = y \) and we know the answer: \( y = ce^x \).

The method is essentially undetermined coefficients. We let

\[
y = \sum_{n=0}^{\infty} a_n x^n
\]  

(79)

In order to avoid confusion redefining the lower limit of summation, as we shift the index of summation, it is strongly advised (not done in book) to write the series as a sum indexed from
negative infinity to positive infinity with the constraint that the coefficients corresponding
to the negative indices are zero.

\[ y = \sum_{n=-\infty}^{\infty} a_n x^n, \quad a_n = 0, \text{ when } n < 0. \]  

(80)

Then

\[ y' = \sum_{m=-\infty}^{\infty} a_m m x^{m-1} \]  

(81)

Notice that we changed the dummy index of summation, from \( n \) to \( m \) (we did the differentia-
tion in the formula with \( n \) and then switched from \( n \) to \( m \)). We insert the PS expansions
for \( y \) and \( y' \) in the ODE and obtain

\[ \sum_{m=-\infty}^{\infty} a_m m x^{m-1} = \sum_{n=-\infty}^{\infty} a_n x^n. \]  

(82)

In order to match coefficients comfortably, we need to match exponents. We let \( m = n + 1 \).

\[ \sum_{n=-\infty}^{\infty} a_{n+1} (n + 1) x^n = \sum_{n=-\infty}^{\infty} a_n x^n. \]  

(83)

Matching the coefficients and recalling the constraint on negative indices, we obtain,

\[ a_{n+1} (n + 1) = a_n, \text{ for all } n; \text{ coefficients with negative indices equal zero.} \]  

(84)

When \( n < -1 \), the condition is automatically satisfied; both \( n \) and \( n + 1 \) are negative. When
\( n = -1 \), the condition is again automatically satisfied since \( n + 1 = 0 \). When \( n \geq 0 \),

\[ a_{n+1} = \frac{a_n}{n + 1} \]  

(85)

This is a recursive formula that obtains the coefficient of \( x^{n+1} \) in the original expansion fo \( y \)
in terms of the coefficient of \( x^n \). We obtain easily

\[ a_n = \frac{a_0}{1 \cdot 2 \cdot 3 \cdots n} = \frac{a_0}{n!} \]  

(86)

Thus,

\[ y(x) = a_0 \left( \sum_{n=0}^{\infty} \frac{x^n}{n!} \right), \]  

(87)

where \( a_0 \) is arbitrary. This is the Taylor expansion of the function \( a_0 e^x \).

**Example 2: The Airy equation \( y'' - xy = 0 \)**

We apply the above procedure to obtain series representations of the solutions of the Airy equation

\[ y'' - xy = 0, \]  

(88)

which is not solvable in closed form. All our summations are from negative to positive infinity. The subscript under the summation sign indicates the index of summation. We seek a power series solution,

\[ y = \sum_n a_n x^n, \quad \text{all coefficients with negative index are zero.} \]  

(89)

\[ y'' = \sum_n n(n-1)a_n x^{n-2}, \]  

(90)

Inserting the series for \( y \) and \( y'' \) in the ODE, we obtain,

\[ \sum_m m(m-1)a_m x^{m-2} - \sum_n a_n x^{n+1} = 0, \]  

(91)

We let \( m = k + 2 \) and \( n = k - 1 \), so that the power of \( x \) is \( x^k \) in both summations. We then place both terms under a single summation sign.

\[ \sum_k ((k + 2)(k + 1)a_{k+2} - a_{k-1})x^k = 0, \quad \text{for all integers } k. \]  

(92)
Necessarily,

\[(k + 2)(k + 1)a_{k+2} - a_{k-1} = 0, \quad \text{for all integers } k.\]  \hspace{1cm} (93)

It is helpful to shift the index so that the largest index is \(k\).

\[k(k - 1)a_k - a_{k-3} = 0, \quad \text{for all integers } k.\] \hspace{1cm} (94)

- When \(k < 0\), both indices \(k\) and \(k - 3\) are negative. Thus, both \(a_k\) and \(a_{k-3}\) are zero. Condition (94) is satisfied.
- When \(k = 0\) and when \(k = 1\), condition (94) is again satisfied (check).
- For larger \(k \geq 2\), we solve relation(94) for \(a_k\),

\[a_k = \frac{a_{k-3}}{k(k - 1)}, \quad k = 2, 3, 4, \ldots.\]  \hspace{1cm} (95)

- By setting \(k = 2\) and recalling that \(a_{-1} = 0\), we see that \(a_2\) is zero and consequently \(a_5, a_8, a_{11}a_{14}, \ldots\) are all zero.
- By setting \(k = 3, 6, 9, \ldots\), we obtain

\[a_3 = \frac{a_0}{3 \cdot 2}, \quad a_6 = \frac{a_3}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}, \quad a_9 = \frac{a_6}{9 \cdot 8} = \frac{a_0}{9 \cdot 8 \cdot 6 \cdot 5 \cdot 3 \cdot 2}, \quad \ldots\] \hspace{1cm} (96)

- Similarly by setting \(k = 4, 7, 10, \ldots\), we obtain

\[a_4 = \frac{a_1}{4 \cdot 3}, \quad a_7 = \frac{a_4}{7 \cdot 6 \cdot 4 \cdot 3}, \quad a_{10} = \frac{a_1}{10 \cdot 9 \cdot 7 \cdot 6 \cdot 4 \cdot 3}, \quad \ldots\] \hspace{1cm} (97)

The coefficients \(a_0\) and \(a_1\) can be chosen arbitrarily.
The general solution of the Airy equation is

\[
y(x) = a_0 \left[ 1 + \frac{x^3}{2 \cdot 3} + \frac{x^6}{2 \cdot 3 \cdot 5 \cdot 6} + \frac{x^{3n}}{2 \cdot 3 \cdots (3n-1)(3n) \cdots} + \cdots \right]
\]

\[
y_1(x)
\]

+ \[
a_1 \left[ x + \frac{x^4}{3 \cdot 4} + \frac{x^7}{3 \cdot 4 \cdot 6 \cdot 7} + \frac{x^{3n+1}}{3 \cdot 4 \cdots (3n)(3n+1) + \cdots} \right]
\]

\[
y_2(x)
\]

The series \(y_1(x)\) and \(y_2(x)\) form a basis of the space of solutions of the Airy equation. Both series have an infinite radius of convergence (verify).

In the general case of the ODE \(y'' + p(x)y' + q(x)y = 0\), in which the functions \(p(x)\) and \(q(x)\) are analytic at \(x = 0\) (have power series expansion centered at \(x = 0\) with positive radius of convergence) we insert the series for \(y\), \(y'\), \(y''\), \(p\) and \(q\) into the equation and we proceed similarly to obtain the coefficients in the power series of \(y\) recursively. Such points are called **ordinary points**.

**Singular points:** The point \(x = 0\) is a **regular singular point** of the ODE \(y'' + p(x)y' + q(x)y = 0\), if at least one of the functions \(p(x)\) and \(q(x)\) is not analytic at \(x = 0\), while the functions \(xp(x)\) and \(x^2q(x)\) are analytic. Thus, the functions \(p(x)\) and \(q(x)\) have form

\[
p(x) = \frac{1}{x} (p_0 + p_1 x + p_2 x^2 + p_3 x^3 + \cdots), \quad q(x) = \frac{1}{x^2} (q_0 + q_1 x + q_2 x^2 + q_3 x^3 + \cdots),
\]

where at least one of the numbers \(p_0\), \(q_0\) and \(q_1\) is nonzero. Notice that if \(p_0 = 0\), the function \(p(x)\) is analytic due to cancelation. The same outcome occurs with the function \(q(x)\), if \(q_0 = 0\) and \(q_1 = 0\). For regular singular points, the right "guess" for the solution is

\[
y = x^r (a_0 + a_1 x + a_2 x^2 + \cdots + \cdots), \quad a_0 \neq 0.
\]

(100)
When the guess is inserted in the equation, the lowest power of $x$ obtained is $x^{r-2}$. It comes with coefficients

1. $a_0 r (r - 1) x^{r-2}$, from the term $y''$ of the ODE,

2. $a_0 p_0 r$, from the term $p(x)y'$,

3. $a_0 q_0$, from the term $q(x)y$.

The sum of the three coefficients must equal zero. We obtain the quadratic equation, known as the **indicial** equation,

$$r(r - 1) + p_0 r + q_0 = 0 \quad (101)$$

from which $r$ is calculated. In general there are two roots.

The general solution is obtained by matching coefficients in all the powers of the series. There is considerable subtlety in the cases of equal roots and roots differing by an integer.

**Important remark:** For a series solution of a differential equation to be fully acceptable as a function, the series must have a nonzero radius of convergence and thus represent a function. It often happens that a series solution, obtained by matching coefficients, has radius of convergence equal to zero, that is, the series does not converge for $x \neq 0$. *Asymptotic series* which go beyond the scope of these notes, fall in this category. Such series are still valuable in science. Truncations of such series still provide approximations of a solution, as $x$ approaches zero. There is a catch though. As more terms of the series are included in the truncation, the approximation is improved inside of a *shrinking* domain of the variable $x$ near zero. Typically, however, the approximation becomes dramatically worse as $x$ moves outside away from the window. Thus, there is a trade-off between the degree of accuracy and the extent of the domain of validity of this accuracy. Asymptotic series are in abundance in science and engineering.
Initial Value problems: The Laplace Transform

Initial value problems often have time as the independent variable. We, thus, switch notation from \( x \) to \( t \). The Laplace transform is a linear operator \( \mathcal{L} \) acting on functions \( f(t) \) that are defined for \( t > 0 \). The output of \( \mathcal{L} \) is a function of a different variable \( s \), given by the formula

\[
\mathcal{L} f = \int_0^\infty e^{-st} f(t) \, dt = F(s).
\]  

(102)

For example, if \( f(t) \) equals the constant 1, the output \( \mathcal{L} f \) equals the function \( \frac{1}{s} \). The output function \( F(s) \) is referred to as the *Laplace transform of* \( f(t) \). The function \( f(t) \) is the inverse Laplace transform of the function \( F(s) \).

There is an integral formula for the inverse Laplace transform that produces \( f(t) \), given \( F(s) \). The formula utilizes integration in the complex plane. Often in practice (and always in this class) one obtains the inverse transform from tables (e.g. table 6.2.1 in the book). The student should verify the correctness of the entries in items 1-12 of the table by performing the integrations in formula (102) for the appropriate functions \( f \).

**Properties of the Laplace transform:**  (verify)

1. It is a *linear operator*

2. It transforms *differentiation* of the input function to *multiplication* in the transform in the following sense.

\[
\mathcal{L} f'(t) = sF(s) - f(0).
\]  

(103)

One derives this by applying equation (102) to \( f' \) and integrating by parts in order to recover \( f \) from \( f' \) inside the integral. We find

\[
\mathcal{L} f''(t) = s^2F(s) - sf(0) - f'(0),
\]  

(104)

by applying \( \mathcal{L} \) to \( f''(t) \) and using the above property applied to \( f'(t) \). The transforms of \( f''' \), and higher derivatives are obtained in a similar way in terms of the transform of
Use of the transform

The Laplace transform facilitates the solution of initial value problems for linear ODE with constant coefficients. The simple example below is easier to solve without the use of the transform, but it demonstrates how the transform is applied to ODE.

\[
\frac{dy}{dt} = y + 1, \quad y(0) = 2
\] (105)

Apply \(\mathcal{L}\) on both sides and utilize its linearity to distribute on the right side of the equation.

\[
\mathcal{L}\frac{dy}{dt} = \mathcal{L}(y + 1), \quad \mathcal{L}\frac{dy}{dt} = \mathcal{L}y + \mathcal{L}1, \quad sY - 2 = Y + \frac{1}{s},
\] (106)

Solve for \(Y\) and utilize partial fractions,

\[
Y = \frac{3}{s - 1} - \frac{1}{s}
\] (107)

Apply the inverse transform on both sides and use linearity to distribute,

\[
\mathcal{L}^{-1}Y = \mathcal{L}^{-1}\frac{3}{s - 1} - \mathcal{L}^{-1}\frac{1}{s}
\] (108)

Take the inverses, using the transform table,

\[
y = 3e^t - 1
\] (109)

The function \(y(t)\) obtained satisfies the initial value problem (105).

The low cut-off (Heaviside) function \(u_c(t)\), the Dirac delta function \(\delta(t - c)\), and the convolution \(f \ast g\) of two functions, \(f\) and \(g\).
It turns out that the function

\[ u_c(t) = \begin{cases} 
1 & \text{when } x > c, \\
0 & \text{when } x < c,
\end{cases} \quad (110) \]

is useful when taking inverses of Laplace transforms, mainly because of item 13 in the table 6.2.1. We discuss items 13-19 of table 6.2.1 below. As mentioned earlier, Items 1-12 give the transforms of some classical functions.

- Item 13: Notice that \( u_c(t) f(t - c) \) in this item is \( f(t) \) **shifted** by \( c \). Why the factor \( u_c(t) \)? The Laplace transform is applied to functions whose domain is the *positive* real half-axis; in its defining integral, the transform integrates from *zero* to infinity. Equivalently, we can perform the integration, from negative infinity to positive infinity, while at the same time “cutting-off” the function at negative values of \( t \), in other words, making the function equal to zero at these values. If the function that has been thus cut off, is shifted to the right by the amount \( c > 0 \), the shifted function will be zero in the interval \((0, c)\). If the function in question is, for example, \( \cos t \), the shifted function will be \( u_c(t) \cos(t - c) \). The importance of the shift in the Laplace transform comes from item 13 of the table: *in order to calculate the inverse transform of the product of the exponential \( e^{-cs} \) and a function \( F(s) \) with known inverse transform \( f(t) \), all we need to do is shift \( f \) by the amount \( c \).* In terms of formulae,

\[ \mathcal{L}^{-1} \left( e^{-cs} F(s) \right) = f(t - s). \quad (111) \]

- Item 17: The Dirac delta function \( \delta(t) \), introduced by Dirac in the early nineteen hundreds, is an idealization, that can be thought of as the derivative of the Heaviside
function \( H(t) \) (identical to our \( u_0(t) \)),

\[
H(t) = \begin{cases} 
1 & \text{when } t > 0, \\
0 & \text{when } t < 0.
\end{cases} \tag{112}
\]

The derivative of \( H(t) \) is clearly zero at all \( t \neq 0 \). It is infinite at \( t = 0 \), since \( H(t) \) exhibits infinite slope at this point. Thus, the delta function cannot be defined as a bona fide function. It is rather a limiting object in the following sense. Imagine an approximation of the Heaviside function, in which the passage from the value zero to the value 1 at \( t = 0 \) does not occur instantaneously, but over a tiny interval \( \Delta t \), that contains the point \( t = 0 \). Its derivative, let us call it \( \delta_{\text{approx}}(t) \), is zero outside the tiny interval and achieves high values in the interval. Moreover,

\[
\int \delta_{\text{approx}}(t) \, dt = 1, \quad \text{integral taken over the interval } \Delta t. \tag{113}
\]

the value of the integral being equal to the jump in the approximate Heaviside function from zero to 1. The delta function is a limit in some sense of all possible \( \delta_{\text{approx}}(t) \) as \( \Delta t \to 0 \). The delta function is is incorporated in rigorous mathematics as an object called “distribution”. Nevertheless, it is widely written as a function, with the exercise of proper care. Clearly (explain), the delta function satisfies,

\[
\int_{-\infty}^{\infty} \delta(t) f(t) \, dt = f(0), \quad \int_{-\infty}^{\infty} \delta(t - c) f(t) \, dt = f(c), \tag{114}
\]

that hold for every continuous function \( f(t) \). Thus, the Laplace transform of \( \delta(t - c) \), the shifted delta function with \( c > 0 \), is the exponential \( e^{-cs} \).

- item 16: A new operator is introduced here, the **convolution operator**. Its input consists of two functions \( f(t) \) and \( g(t) \). Its output, a function, say \( k(t) \), is given by the
The convolution of $f$ and $g$ is denoted symbolically by $f \ast g$. Item 16 in table 6.2.1 of the book reveals the importance of the convolution:

$\mathcal{L}(f \ast g) = F(s)G(s).$  \hfill (116)

The inverse Laplace transform transforms multiplication to convolution. Thus, when inverting a product of two factors, we only need to invert each factor and take the convolution of the two results. This benefit extends to more factors, as we see below. Moreover, the convolution operation inherits basic algebraic properties from the multiplication operation. Thus, the convolution is associative, commutative and distributive,

$$(f \ast g) \ast h = f \ast (g \ast h), \quad f \ast g = g \ast f, \quad f \ast (g + h) = f \ast g + f \ast h.$$  \hfill (117)

One may verify these properties directly by using the definition of the convolution and manipulating integrals.

- Item 14 addresses the effect of shifting the transform $F(s)$
- Item 15 addresses the effect of scaling the transform
- Item 18 addresses the effect of taking $n$ derivatives of $f(t)$
- Item 19 addresses the effect of taking $n$ derivatives of the transform
Boundary Conditions

It occurs often in practice, that a differential equation holds in a bounded domain of the independent variable. Our interest at the moment is the ODE

\[ y'' + p(x)y' + q(x)y = f(x), \quad x_0 < x < x_1, \tag{118} \]

holding on a given open interval \((x_0, x_1)\). The general solution of (118) has form,

\[ y = c_1 y_1(x) + c_2 y_2(x), \tag{119} \]

where \(c_1\) and \(c_2\) are arbitrary constants. What determines the constants, in applications, is some given information of the solution at the boundary of the interval, that is at points \(x_0\) and \(x_1\). Such information is provided by so-called boundary conditions. The following three types of linear boundary conditions are commonly encountered.

1. \(y = c\) (Dirichlet boundary condition)
2. \(y' = c\) (Neumann boundary condition)
3. \(y' + cy = c\) (Robin, or third kind, boundary condition).

In each boundary condition, \(c\) is a given constant. If \(c = 0\), the boundary condition is homogeneous. When \(c \neq 0\), the boundary condition is nonhomogeneous. The boundary value problem (BVP), that is the ODE and the boundary conditions taken together, is linear if both the ODE and the boundary conditions are linear. In addition, the BVP is homogeneous, if both the ODE and the boundary conditions are homogeneous.

For the above ODE, a total of two boundary conditions are required, one at each boundary point. They may be of the same type (e.g. both Dirichlet) or of different types. The two boundary conditions, taken together, form a linear system of algebraic equations for the determination of the two constants \(c_1\) and \(c_2\). The two constants are determined uniquely,
if the coefficient matrix of the system is invertible, that is if its determinant is nonzero. If
the determinant is zero, there are two possible outcomes.

- the system is inconsistent, thus, the BVP has no solution;
- the two equations of the system are linearly dependent, thus, only one of the two is
  kept. It has the form \( a_1c_1 + a_2c_2 = b \), where \( a_1, a_2, b \) are scalars. The BVP has
  infinitely many solutions in this case. If the BVP is homogeneous, \( a_1c_1 + a_2c_2 = 0 \).

For example, consider the simplest case of the ODE

\[
y'' = 0, \tag{120}
\]

in the interval \( 0 < x < L \). According to an example considered early in these notes, the
function \( y(x) \) represents the equilibrium temperature distribution along a rod that extends
from point 0 to point \( L \), with no thermal loss or gain transversly to it. The general solution
of the ODE is

\[
y = c_1 + c_2x, \tag{121}
\]

Additional physical information is needed to determine the constants \( c_1 \) and \( c_2 \). Let us
provide such information. We stipulate \( y(0) = y_0 \) and \( y'(L) = y_1 \) (Dirichlet boundary
conditions) The system of boundary conditions is

\[
\begin{cases} 
  c_1 = y_0 \\
  c_1 + c_2L = y_1,
\end{cases}
\]

with the unique solution

\[
\begin{cases} 
  c_1 = y_0 \\
  c_2 = \frac{y_1 - y_0}{L}.
\end{cases}
\tag{122}
\]

The equilibrium temperature in the rod varies linearly along the rod.

We now change the boundary condition at \( x + L \) to the homogeneous Neumann boundary
condition, \( y'(L) = 0 \). What does this mean physically? The heat flux along the rod is known
to be proportional to the negative temperature gradient. Thus, this condition imposes no
flow of heat in or out of the rod at its end \( x = L \). The system of boundary conditions,

\[
\begin{align*}
    c_1 &= y_0, \\
    c_2 &= 0,
\end{align*}
\]

is already in solved form. It obtains that the equilibrium temperature of a rod, thermally insulated at the one end (also thermally insulated on its side all around) and having fixed temperature \( y_0 \) at the other end is constant along the rod and equal to the value \( y_0 \). The result agrees with the physical intuition that heat flows into and/or out of the rod at the point of fixed temperature, until the temperature \( y = y_0 \) is reached at all points of the rod.

Let us now again alter the boundary conditions to be of homogeneous Neumann type at both ends. The system of boundary conditions is

\[
\begin{align*}
    c_2 &= 0, \\
    c_2 &= 0,
\end{align*}
\]

Heat cannot flow into or out of the rod. Again there is agreement with physical intuition. There is heat flow inside the rod, the temperature of the rod tending to a uniform equilibrium value. What this value is cannot be determined by the BVP. It depends on the amount of heat energy trapped inside the rod.

Let us finally consider the boundary conditions to be of nonhomogeneous Neumann type at both ends with different values \( y' = 1 \) and 2 at points \( x = 0 \) and \( L \) respectively. We obtain the system

\[
\begin{align*}
    c_2 &= 1, \\
    c_2 &= 2,
\end{align*}
\]

The system has no solution. (125)

Recalling that the heat flux along the rod is proportional to the negative temperature gradient, we observe that the heat flux into the rod at \( x = L \) is twice the heat flux out of the rod at \( x = 0 \). As a constant positive net of energy is entering the rod, temperatures in the
rod should be rising. No equilibrium temperature distribution is possible.

**Periodic and antiperiodic boundary condition**

An important type of linear boundary conditions involves both boundary points. Most important among these boundary conditions are

1. \( y(x_0) = y(x_1) \) and \( y'(x_0) = y'(x_1) \) (periodic condition)

2. \( y(x_0) = y(x_1) \) and \( y'(x_0) = -y'(x_1) \) (antiperiodic condition)

In the first case, the solution may be extended to the whole line as a periodic function of period \( x_1 - x_0 \) (how?). In the second case, the solution may be extended to the whole line as a periodic function of period \( 2(x_1 - x_0) \) (how?). Clearly, both the periodic and antiperiodic conditions are homogeneous.

**Summary**

We have the general solution of the ODE (118) that contains two constants to be determined and we have two equations that the boundary conditions provide us with. We consider only linear boundary conditions. Inserting the general solution into the two boundary conditions, we obtain a linear system of the form

\[
\begin{align*}
    a_{11}c_1 + a_{12}c_2 &= b_1 \\
    a_{21}c_1 + a_{22}c_2 &= b_2
\end{align*}
\]

(126)

If the determinant of the coefficients \( a_{ij} \) is nonzero, the system has a unique solution. The case of a zero determinant plays a pivotal role in the analysis of boundary value eigenvalue problems that follow.
Boundary Value Eigenvalue Problems

We regard *homogeneous* boundary conditions as a part of our operator $\frac{d^2}{dx^2} + p(x)\frac{d}{dx} + q(x)$. This makes sense, since the boundary conditions *participate* in the definition of the domain of the operator (functions that are twice differentiable and satisfy the boundary conditions). We follow this point of view for *only homogeneous* boundary conditions. For such conditions, the domain of the operator is a *vector space*, something that fails to happen, if at least one of the boundary conditions is nonhomogeneous (why?).

We tackle the problem of solving the eigenvalue problem for second order linear differential operators. We begin with the simple case of the operator

$$\mathcal{L} = -\frac{d^2}{dx^2}, \quad 0 < x < L,$$  \hfill (127)

acting on functions $u(x)$ that are twice differentiable in the interval $(0, L)$ and satisfy *Dirichlet* boundary conditions,

$$u(0) = 0, \quad u(L) = 0.$$  \hfill (128)

The eigenvalue equation is

$$\mathcal{L}u = \lambda u, \quad u'' + \lambda u = 0.$$  \hfill (129)

Its general solution is

$$\begin{cases} 
  u(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x), & \text{if } \lambda \neq 0 \\
  c_1 + c_2 x, & \text{if } \lambda = 0.
\end{cases}$$  \hfill (130)

In the case $\lambda = 0$, the boundary conditions imply $c_1 = 0$ and $c_2 = 0$. Thus, $\lambda = 0$ is *not an eigenvalue*. 
Applying the boundary conditions to the general solution when $\lambda \neq 0$ gives

$$\begin{cases} u(0) = c_1 = 0, \\ u(L) = c_1 \cos(\sqrt{\lambda}L) + c_2 \sin(\sqrt{\lambda}L) = c_2 \sin(\sqrt{\lambda}L) = 0. \end{cases}$$ (131)

We thus obtain the eigenvalue condition,

$$\sin(\sqrt{\lambda}L) = 0.$$ (132)

This equation is the analogue of the characteristic polynomial equation of matrix eigenvalue problems. Instead of a polynomial though, which has $n$ roots when it comes form an $n \times n$ matrix, the eigenvalue condition involves the sine function, which produces infinitely many eigenvalues, $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n, \cdots$ and corresponding eigenvectors $\psi_1, \psi_2, \psi_3, \cdots, \psi_n, \cdots$. Indeed, the values of $\lambda$, that satisfy the eigenvalue condition are

$$\sqrt{\lambda}L = n\pi, \quad n = 1, 2, 3, \cdots$$ (133)

from which it follows that

**Dirichlet BC:** $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\psi_n(x) = \sin(\sqrt{\lambda_n}x) = \sin \frac{n\pi x}{L}$, $n = 1, 2, 3, \cdots$ (134)

A similar calculation for the same operator $(-\frac{d^2}{dx^2})$, but with **Neumann boundary conditions** at both endpoints produces the result,

**Neumann BC:** $\lambda_0 = 0$, $\psi_0(x) = 1; \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\psi_n(x) = \cos(\sqrt{\lambda_n}x) = \cos \frac{n\pi x}{L}$, $n = 1, 2, 3, \cdots$ (135)

Observe that $\lambda = 0$ is an eigenvalue of the Neumann problem with the constant function as the corresponding eigenfunction. In other word, the operator with Neumann boundary conditions has a nontrivial nullspace. Consequently, the operator is not invertible in this
case. It is good practice for the student to perform this calculation with a Dirichlet condition at one end and a Neumann condition on the other.

We now tackle the eigenvalue problem with periodic boundary conditions. The operator is still the negative second derivative. It is convenient to label the interval in which our functions $u(x)$ are defined to be $(-L, L)$ instead of the $(0, L)$, that we used previously.

$$\mathbb{L} = -\frac{d^2}{dx^2}, \quad 0 < x < 2L \text{ with BC: } u(-L) = u(L) \text{ and } u'(-L) = u'(L).$$

(136)

The result is in this case,

<table>
<thead>
<tr>
<th>periodic BC:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_0 = 0, ; \psi_0(x) = 1; ; \lambda_n = \left(\frac{n\pi}{L}\right)^2, ; \psi_n^{(1)}(x) = \cos \frac{n\pi x}{L}, ; \psi_n^{(2)}(x) = \sin \frac{n\pi x}{L}, ; n = 1, 2, 3, \cdots$</td>
</tr>
</tbody>
</table>

(137)

• The eigenvalues for $n = 1, 2, 3, \cdots$ are double. To each correspond two linearly independent eigenfunctions, $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$.

• There is of course an infinity of choices for a basis of the eigenspace of a double eigenvalue. A common choice are the cosine and sine functions above. An alternative choice, that is sometimes more convenient, makes use of the two complex exponential solutions:

<table>
<thead>
<tr>
<th>periodic BC:</th>
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<tbody>
<tr>
<td>$\lambda_0 = 0, ; \psi_0(x) = 1; ; \lambda_n = \left(\frac{n\pi}{L}\right)^2, ; \psi_n(x) = e^{\pm \frac{in\pi x}{L}}, ; n = 1, 2, 3, \cdots$</td>
</tr>
</tbody>
</table>

(138)

The two basis eigenfunctions are now $e^{\frac{in\pi x}{L}}$ and $e^{-\frac{in\pi x}{L}}$.

• The student should solve the eigenvalue problem going through all steps, making certain that an eigenvalue and/or eigenfunction have not been missed.
Inner product and orthogonality

It is often advantageous to work within the vector space of functions $f(x)$, that are defined on some interval $(a, b)$ and have the property that

$$\int_a^b |f(x)|^2 dx < \infty.$$  \hspace{1cm} (139)

These functions are referred to as **square-integrable on the interval** $(a, b)$. The space (set) of functions is labeled $L^2(a, b)$. The space includes functions that cannot be integrated in the classic sense taught in calculus (Riemann integration), but can be integrated in a more subtle sense (Lebesgue integration). Since all the functions that we will encounter explicitly are Riemann integrable, we need not dwell on this issue.

We want to work with functions in $L^2(a, b)$ in a way that is similar to how we work with vectors in $\mathbb{R}^n$. In analogy to the dot product of two vectors with real entries $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i,$$ \hspace{1cm} (140)

we define the **inner product** of two real valued functions $f(x)$ and $g(x)$ in $L^2(a, b)$ by

$$(f, g) = \int_a^b f(x)g(x)dx.$$ \hspace{1cm} (141)

Same as the dot product of two vectors, the inner product of two functions is a scalar. Notice that the integral here plays the role of the sum in the dot product. The variable of integration $x$ plays the role of the index of summation $i$. In analogy to the dot product, we say that two functions are **orthogonal** when their inner product equals zero.

**Reality of eigenvalues; the eigenfunctions as an orthogonal basis of the space $L^2$:**

The eigenvalues and eigenfunctions of the problem with zero Dirichlet boundary conditions have the following important properties.

1. All eigenvalues are real. We show this below.
2. Any two distinct eigenfunctions are orthogonal to each other. In other words, their inner product is zero. This is easily verified by calculating the corresponding integrals.

3. The set of eigenfunctions spans the space $L^2$ over the corresponding interval. In other words, each $L^2$ function $f$ (generally, complex valued) can be represented as a finite or infinite linear superposition of the eigenfunctions. The proof of this statement is somewhat involved and we skip it.

4. The set of eigenfunctions constitutes a basis of the space $L^2$ over the corresponding interval. This statement follows from the first two and the definition of a basis.

Exactly the same properties hold for the eigenvalues and eigenvectors of the problems with (a) zero Neumann boundary conditions, (b) zero Dirichlet boundary conditions at one endpoint and zero Neumann boundary conditions at the other, (c) periodic boundary conditions), and (d) many others.

Before we explain how these phenomena follow from a property of some linear differential operators called self-adjointness, we expand on the idea of of a space of functions.

Let us first push the the analogies between $\mathbb{R}^n$ and $L^2(0, L)$.

1. Orthogonality

   (a) **Vectors in $\mathbb{R}^n$**: Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are orthogonal if their dot product is zero, $\mathbf{u} \cdot \mathbf{v} = 0$.

   (b) **Functions in $L^2$**: Two functions $f$ and $g$ in $L^2(0, L)$ are orthogonal if their inner product is zero, $(f, g) = 0$.

2. Representation as a linear superposition of basis vectors.

   (a) **Vectors in $\mathbb{R}^n$**: Let $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \cdots, \mathbf{b}_n$ be an orthogonal basis of $\mathbb{R}^n$. A vector $\mathbf{v}$ is represented as a linear combination of basis vectors,

   $$\mathbf{v} = c_1 \mathbf{b}_1 + c_2 \mathbf{b}_2 + c_3 \mathbf{b}_3 + \cdots + c_n \mathbf{b}_n.$$ 

   (142)
By dotting the equality by \( b_k \) for any index \( k = 1, 2, 3, \cdots, n \), and implementing orthogonality (\( b_k \cdot b_i = 0 \) if \( k \neq i \)), we obtain a simple formula for the coefficient \( c_k \).

\[
c_k = \frac{v \cdot b_k}{b_k \cdot b_k}. \tag{143}
\]

(b) **Functions in \( L^2 \):** Similarly, let \( \psi_1, \psi_2, \psi_3, \cdots, \psi_n, \cdots \) be an orthogonal basis of \( L^2(0, L) \). A function \( f(x) \) is represented as a finite or infinite linear superposition of basis functions, that is,

\[
f(x) = c_1\psi_1(x) + c_2\psi_2(x) + c_3\psi_3(x) + \cdots. \tag{144}
\]

We may take the inner product of both sides of the equation with \( \psi_k(x) \) for any index \( k = 1, 2, 3 \cdots \) (this means multiplying the equation by \( \psi_k(x) \), then integrating from 0 to \( L \)). Again, implementing orthogonality (\( (\psi_k, \psi_i) = 0 \) if \( k \neq i \)), we obtain a simple formula for the coefficient \( c_k \).

\[
c_k = \frac{(f, \psi_k)}{(\psi_k, \psi_k)} = \frac{\int_0^L f(x)\psi(x)dx}{\int_0^L \psi^2(x)dx} \tag{145}
\]

3. Magnitude or norm.

(a) **Vectors in \( \mathbb{R}^n \):** We say that the scalar \( \sqrt{v \cdot v} \) is a measure of the magnitude or length of the vector \( v \) and we denote it by \( |v| \) or \( \|v\| \).

(b) **Functions in \( L^2 \):** We call the corresponding quantity \( \sqrt{(f, f)} \) a norm of the function \( f(x) \) and we denote it by \( \|f\| \).

**Generalization:** (optional)

- A rule that assigns to each element of a vector space a positive or zero value is said to be a norm if it has the following three properties.

1. \( \|cf\| = |c|\|f\| \), for any scalar \( c \).
2. \( \|f + g\| \leq \|f\| + \|g\| \) for any pair of functions \( f \) and \( g \) (triangle inequality).

3. If \( \|f\| = 0 \), then \( f(x) \equiv 0 \).

A norm defines the space of all functions to which it assigns a finite value. For example, the space \( C[0, L] \) is the set of all functions that are continuous on the closed set \([0, 1]\), with norm

\[
\|f\| = \max_{0 \leq x \leq L} |f(x)|.
\]

Thus, a norm may be defined with the aid of an inner product as happens with the \( L^2 \) norm, but this is not necessary.

- As with norms, inner products can be defined more generally. The only need to satisfy a small number of basic properties. We will work mainly with the inner product defined above.

**Complex valued functions and complex inner product:** We broaden our definition of the inner product to *include complex valued functions*. We have already encountered such functions as the exponential basis functions when we apply periodic boundary conditions. The inner product of two functions \( f(x) \) and \( g(x) \) in \( L^2(a, b) \), that are real or complex is given by

\[
(f, g) = \int_a^b f(x)\overline{g(x)}dx,
\]

where the line over \( g(x) \) indicates the complex conjugate of \( g(x) \). In this way, the norm of a function \( f(x) \)

\[
\|f\| = (f, f)
\]

remains a number that is positive or zero. The inner product has the following properties.

1. \( (f, g) = (g, f) \)

2. \( (c_1f_1 + c_2f_2, g) = c_1(f_1, g) + c_2(f_2, g) \)

3. If \( f \) is not identically equal to zero, then \( (f, f) \) is a real positive number \((f, f) > 0\).
Exercise: Using only the three properties, prove that

1. If $f$ is identically equal to zero, then $(f, g) = 0$, for every $g$ in $L^2$.

2. $(f, c_1g_1 + c_2g_2) = c_1(f, g_1) + c_2(f, g_2)$

It is convenient to normalize the basis vectors to have norm equal to 1 (think of the basis $i, j, k$, of vector calculus). The normalized basis vector corresponding to an eigenvactor $\psi_n(x)$ is

$$\phi_n(x) = \frac{\psi_n(x)}{\|\psi_n\|}.$$  \hspace{1cm}(149)

Clearly, $\|\phi_n\| = \|\psi_n\| = 1$

**Selfadjointness**

In all three cases of boundary conditions considered, the operator

$$\mathbb{L} = -\frac{d^2}{dx^2},$$  \hspace{1cm}(150)

satisfies the condition, that for all functions $u(x)$ and $v(x)$ in its domain,

$$(\mathbb{L}u, v) = (u, \mathbb{L}v).$$  \hspace{1cm}(151)

The proof of this relation is based on calculating the derivative of the Wronskian of the functions $u(x)$ and $\bar{v}(x)$, the complex conjugate of $v(x)$.

$$\begin{vmatrix}
\frac{d}{dx} u(x) & \bar{v}(x) \\
u'(x) & \bar{v}'(x)
\end{vmatrix} = \begin{vmatrix}
u(x) & \bar{v}(x) \\
u''(x) & \bar{v}'(x)
\end{vmatrix} = u(x)\bar{v}''(x) - u''(x)\bar{v}(x).$$  \hspace{1cm}(152)

Integrating this relation over the interval (say $(a, b)$) over which the functions in the domain
of the operator are defined, we obtain

\[
\begin{vmatrix}
  u(b) & \bar{v}(b) \\
  u'(b) & \bar{v}'(b) \\
  u(a) & \bar{v}(a) \\
  u'(a) & \bar{v}'(a)
\end{vmatrix}
- \begin{vmatrix}
  u(a) & \bar{v}(a) \\
  u'(a) & \bar{v}'(a)
\end{vmatrix}
= \int_a^b u\bar{v}''dx - \int_a^b u''\bar{v}dx = -(u, \mathbb{L}v) + (\mathbb{L}u, v),
\] (153)

where we used the fact that \( \bar{v}'' = \bar{v}'' \), in other words, \( \mathbb{L}\bar{v} = \mathbb{L}v \). Property (151) is satisfied, if and only if the left side of equation (153) is zero. For Dirichlet zero boundary conditions, both determinants have the first row entries equal to zero. For Neumann zero boundary conditions, the second row entries equal zero. Finally, for periodic boundary conditions the two determinants are equal to each other. Thus, in all three cases the left hand side is zero. This proves relation (151). A second order differential operator on a finite interval with boundary conditions, that satisfies relation (151) is said to be selfadjoint. We now prove two basic properties of selfadjoint operators.

1. **Reality of eigenvalues.** Let \( \mathbb{L} \) be a selfadjoint operator, \( \lambda \) an eigenvalue of \( \mathbb{L} \) and \( \psi \) a corresponding \( L^2 \) eigenvector. We have \( \mathbb{L}\psi = \lambda\psi \). Thus,

\[
(\mathbb{L}\psi, \psi) = (\lambda\psi, \psi) = \lambda(\psi, \psi) \quad \text{and} \quad (\psi, \mathbb{L}\psi) = (\psi, \lambda\psi) = \bar{\lambda}(\psi, \psi).
\] (154)

Due to the selfadjointness of \( \mathbb{L} \), the left sides of these equalities are equal to each other, hence the right sides are equal to each other as well.

\[
\lambda(\psi, \psi) - \bar{\lambda}(\psi, \psi) = 0, \quad (\lambda - \bar{\lambda})(\psi, \psi) = 0.
\] (155)

Since \( \psi \) is an eigenvector, \( (\psi, \psi) \neq 0 \). Necessarily, \( \lambda = \bar{\lambda} \), in other words, \( \lambda \) is real.

2. **Orthogonality of \( L^2 \) eigenvectors corresponding to different eigenvalues.** Let \( \mathbb{L} \) be a selfadjoint operator, \( \lambda \) and \( \mu \) distinct eigenvalues of \( \mathbb{L} \) and \( u \) and \( v \) corresponding \( L^2 \) eigenvectors. By the previous result, \( \lambda \) and \( \mu \) are real. We have \( \mathbb{L}u = \lambda u \) and

55
$Lv = \mu v$. Thus,

$$
(Lu, v) = \lambda(u, v) \quad \text{and} \quad (u, Lv) = \mu(u, v). \tag{156}
$$

Due to the selfadjointness of $L$, the left sides of these equalities are equal to each other, hence the right sides are equal to each other as well. Thus,

$$
\lambda(u, v) - \mu(u, v) = 0, \quad (\lambda - \mu)(u, v) = 0. \tag{157}
$$

The assumption $\lambda \neq \mu$ implies that $(u, v) = 0$, the eigenvectors $u$ and $v$ are orthogonal to each other.

Exercise: Under what boundary conditions is the differential operator

$$
-\frac{d}{dx} p(x) \frac{d}{dx} + q(x) \tag{158}
$$

self-adjoint, under the assumption that functions $p$ and $q$ are real valued and $p'$ and $q$ are continuous? Assume separated conditions, that is separate conditions at each endpoint.

**Diagonalization of the operator $L$ (Optional)**

We consider the operator with Dirichlet boundary conditions. We apply the operator $L$ to a function

$$
u(x) = \sum_n c_n \phi_n(x) \tag{159}
$$

and we obtain

$$
L u(x) = L \left( \sum_n c_n \phi_n(x) \right) = \sum_n c_n L \phi_n(x) = \sum_n \lambda_n c_n \phi_n(x). \tag{160}
$$
In words, if \( u(x) \) is represented in the coordinate system defined by the eigenfunctions, as a vector of infinitely many components,

\[
(u_1, u_2, u_3, \cdots)
\]

(161)

the function \( L u \) is represented as the vector,

\[
(\lambda_1 u_1, \lambda_2 u_2, \lambda_3 u_3, \cdots).
\]

(162)

In the coordinate system of the eigenfunction basis, the operator \( L \) acts like a matrix with infinitely many rows and columns which is diagonal.

One may verify that the inner product \( (u, v) \) of two functions \( u(x) \) and \( v(x) \), expressed in terms of the new coordinates is

\[
(u, v) = \sum_n u_n \overline{v}_n,
\]

(163)

and the norm of \( u(x) \) is

\[
\|u\| = \sqrt{\sum_n |u_n|^2}.
\]

(164)

The domain of the operator \( L \) is defined in the new coordinates of the space \( L^2 \) in an easy way. It consists of all the functions \( u(x) \) in \( L^2 \) for which

\[
\sum_n |\lambda_n u_n|^2 < \infty
\]

(165)

and which satisfy the boundary conditions. Since \( \lambda_n \to \infty \) as \( n \to \infty \), it is possible for a function \( u(x) \) that

\[
\sum_n |u_n|^2 < \infty \quad \text{and} \quad \sum_n |\lambda_n u_n|^2 = \infty.
\]

(166)

Such a function is not twice-differentiable in the \( L^2 \) sense.
The heat or diffusion equation

We solve the initial value problem for the forced heat equation with zero Dirichlet boundary conditions and an initial condition. We think of $0 < x < L$ as the space variable and $t$ as the time variable. The unknown function $u(x, t)$ represents the temperature at the spatial point $x$ at time $t$. Alternatively, if the equation describes the diffusion of a certain substance in the interval $(0, L)$, then $u(x, t)$ represents the concentration of the substance. The full mathematical problem is

\[ u_t = Du_{xx} + f(x, t), \quad 0 < x < L, \quad t > 0, \]  

(167)

- zero Dirichlet boundary conditions, $u(0, t) = 0$ and $u(L, t) = 0$,
- initial condition $u(x, 0) = h(x)$.

The function $f(x, t)$ is a source term. The coefficient $D > 0$ is known as the diffusion coefficient.

We rewrite the PDE as

\[ u_t = -D\mathbb{L}u + f(x, t), \quad 0 < x < L, \quad t > 0, \]  

(168)

where $\mathbb{L}$ is the operator $-\frac{d^2}{dx^2}$ with zero Dirichlet boundary conditions. The method of eigenfunction expansions, as it is commonly called, is essentially a method of undetermined coefficients. All functions that involve the variable $x$ are represented as a linear superposition of the eigenfunctions of $\mathbb{L}$. When these functions of space depend on time as well, the coefficients in the series also depend on time. So,

\[ u(x, t) = b_1(t)\psi_1(x) + b_2(t)\psi_2(x) + b_3(t)\psi_3(x) + \cdots, \]  

(169)
The coefficients $b_n(t)$ are the unknowns of the problem. We also have the given

$$u(x, 0) = h(x) = h_1\psi_1(x) + h_2\psi_2(x) + h_3\psi_3(x) + \cdots$$ \hspace{1cm} (170)

The scalar coefficients $h_n$ are calculated with the aid of formula (145). Furthermore,

$$f(x, t) = f_1(t)\psi_1(x) + f_2(t)\psi_2(x) + f_3(t)\psi_3(x) + \cdots$$ \hspace{1cm} (171)

where the functions $f_n(t)$ are again calculated with the aid of formula (145).

We observe that

$$L u = L \{b_1(t)\psi_1(x) + b_2(t)\psi_2(x) + b_3(t)\psi_3(x) + \cdots \}$$

$$= b_1(t)L\psi_1(x) + b_2(t)L\psi_2(x) + b_3(t)L\psi_3(x) + \cdots$$ \hspace{1cm} (172)

$$= \lambda_1 b_1(t)\psi_1(x) + \lambda_2 b_2(t)\psi_2(x) + \lambda_3 b_3(t)\psi_3(x) + \cdots$$

We now insert the expansions for $u$, $Lu$, and $f$ into equation (168) and balance the coefficients of the eigenfunction $\psi_n$ for each index $n$. We obtain a set of first order ODE for the coefficients $b_n(t)$.

$$\frac{d}{dt}b_n = -\lambda_n Db_n + f_n(t), \hspace{0.5cm} n = 1, 2, 3, \cdots \hspace{1cm} (173)$$

These are linear first order ODE, hence, solvable. The eigenvalues $\lambda_n$ have been calculated earlier. For each of the $b_n$, we need the initial condition. We have already obtained this:

$$b_n(0) = h_n, \hspace{0.5cm} n = 1, 2, 3, \cdots \hspace{1cm} (174)$$

Everything in the right side of the solution (169) has now been calculated. The problem is thus solved.
In the simpler, sourceless case $f(x, t) \equiv 0$, the solution of the ODE is immediate.

$$b_n(t) = b_n(0)e^{\lambda_n t}. \quad (175)$$

$$u(x, t) = \sum_{n=1}^{\infty} h_n e^{-\lambda_n Dt} \psi_n(x), \quad \text{where} \quad h_n = \frac{1}{\|\psi_n\|^2} \int_0^L h(s)\psi_n(s) \, ds. \quad (176)$$

We can also manipulate equations (176) further by inserting the expression for $h_n$ in the second equation into the first equation and changing the order of summation and integration.

$$u(x, t) = \int_0^L \left( \sum_{n=1}^{\infty} e^{-\lambda_n Dt} \frac{\psi_n(x)\psi_n(s)}{\|\psi_n\|^2} \right) u(t, 0) \, ds \quad (177)$$

The sum in the parenthesis is the function $G(x, t, s)$ of equation (9). In this equation $T$ our current $u$ and $T_1$ is $u$ at time zero. The sum cannot be calculated explicitly.

Inserting the eigenfunctions and eigenvalues, that correspond to zero Dirichlet boundary conditions, into the first equation (176), one obtains the solution to the BVP

$$u(x, t) = \sum_{n=1}^{\infty} h_n e^{-\left(\frac{n\pi}{L}\right)^2 Dt} \sin \frac{n\pi x}{L}. \quad (178)$$

The solution decays exponentially in time.

In an analogous way, one obtains the solution for Neumann boundary conditions.

$$u(x, t) = \sum_{n=0}^{\infty} h_n e^{-\left(\frac{n\pi}{L}\right)^2 Dt} \cos \frac{n\pi x}{L}. \quad (179)$$

Notice that for $n = 0$, the cosine equals unity and the eigenvalue is zero, in agreement to the values calculated. As $t \to \infty$, the solution converges to the constant solution

$$\lim_{t \to \infty} u(x, t) = h_0 = \frac{1}{L} \int_0^L u(x, 0) \, dx, \quad (180)$$

which is the mean value of the initial temperature distribution.
The solution is obtained in a similar way, for other boundary conditions, for which the problem is selfadjoint.

**Nonhomogeneous boundary conditions:** We let \( u(x, t) = v(x, t) + a(x, t) \) where \( a(x, t) \) is some function that satisfies the boundary conditions required of \( u(x, t) \). As a result \( v(x, t) \) satisfies zero boundary conditions. Inserting the expression for \( u \) into the PDE we obtain a heat equation, for \( v \). The function \( a \) appears in the forcing term.

**Derivation of the heat equation as the model for diffusion:** Exercise. A substance along the \( x \) axis has concentration (mass per unit length) \( u(x, t) \). The flux of the material at point \( x \) (rate at which mass crosses point \( x \) from left to right) is given by the function \( f(x, t) \).

1. Express the time-rate of change of concentration at each point and time, making use only of the function \( u \): \( u_t(x, t) \)

2. Do the same (express the time-rate of change of concentration at each point and time) making use only of the function \( f \). Assume that mass is conserved (neither created nor destroyed). Consider a small interval \( I = (x, x + \Delta x) \) and break the calculation to the following steps.

   (a) time-rate at which mass exits from \( I \) at point \( x + \Delta x \): \( f(x + \Delta x, t) \)

   (b) time-rate at which mass enters \( I \) at point \( x \): \( f(x, t) \)

   (c) time-rate at which nett mass enters \( I \): \( f(x, t) - f(x + \Delta x, t) \)

   (d) time-rate at which average concentration in \( I \) increases: \( \frac{f(x, t) - f(x + \Delta x, t)}{\Delta x} \)

   (e) time-rate at which concentration at point \( x \) increases: \( -f_x(x, t) \).

3. Write an equation that connects the functions \( u \) and \( f \) in the form:
Result from item 1 minus result from item 2 equals zero:

\[ u_t + f_x = 0 \]

4. What would the equation be if mass were created at the rate (mass per unit length per unit time) \( s(x,t) \)?

\[ u_t + f_x = s(x,t) \]

5. Imagine that the substance consists of tiny particles each of which moves around at constant speed, choosing direction randomly with no preference between left and right at a signal from a clock that ticks at every integral multiple of an extremely small time step. Can you suggest a formula for the flux in terms of the concentration?

\[ f = -Du_x, \text{ where } D \text{ is a positive constant.} \]

6. Insert your formula in the first equation you derived: \( u_t = Du_{xx} \)

**The wave equation**

The wave equation in one space dimension is

\[ u_{tt} - c^2u_{xx} = 0. \]  

(181)

One readily verifies that the PDE is satisfied by the expression

\[ u(x,t) = F(x - ct) + G(x + ct), \]  

(182)

where \( F \) and \( G \) are arbitrary functions. Moreover, the above expression gives the general solution of the wave equation on the real spatial line. The left term represents a wave traveling to the right at speed \( c \) (we assume \( c > 0 \)), while the right term represents a wave traveling to the left at speed \( c \). As the wave equation is of second order in time, we need two initial
conditions, \( u(x, 0) \) and \( u_t(x, 0) \). Given

\[
\begin{align*}
  u(x, 0) &= f(x), \\
  u_t(x, 0) &= g(x),
\end{align*}
\]  

we can calculate the functions \( F \) and \( G \). Indeed,

\[
\begin{align*}
  F(x) + G(x) &= f(x), \\
  -cF'(x) + cG'(x) &= g(x).
\end{align*}
\]  

(184)

Taking the derivative of the first equation, leaves us with a system of two equations for the two unknowns \( F'(x) \) and \( G'(x) \). We find

\[
\begin{align*}
  F'(x) &= \frac{1}{2}f'(x) - \frac{1}{2c}g(x), \\
  G'(x) &= \frac{1}{2}f'(x) + \frac{1}{2c}g(x).
\end{align*}
\]  

(185)

We integrate these

\[
\begin{align*}
  F(x) &= \frac{1}{2}f(x) - \frac{1}{2c}h(x), \\
  G(x) &= \frac{1}{2}f(x) + \frac{1}{2c}h(x),
\end{align*}
\]  

(186)

where (a) \( h(x) \) is any antiderivative of \( g(x) \) and (b) the antiderivative of \( f' \) has been chosen to be \( f \), so that the first of equations (184) is satisfied. Inserting these into equation (182), we obtain

\[
\begin{align*}
  u(x, t) &= \frac{1}{2}f(x - ct) - \frac{1}{2c}h(x) + \frac{1}{2}f(x + ct) + \frac{1}{2c}h(x + ct) \\
  &= \frac{1}{2}\{f(x - ct) + f(x + ct)\} + \frac{1}{2c}\{h(x + ct) - h(x - ct)\}.
\end{align*}
\]  

(187)

Recalling that \( h \) is an antiderivative of \( g \), we derive the D’Alambert formula for the solution,

\[
\begin{align*}
  u(x, t) &= \frac{1}{2}\{f(x - ct) + f(x + ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds.
\end{align*}
\]  

(188)
Notice that if $g \equiv 0$, the signal $f(x)$ breaks up into two identical pieces, that travel in opposite directions at speed $c$. It is also instructive to examine the solution when $f \equiv 0$ and $g$ is a delta function centered at $x = 0$.

The wave equation in higher dimensions is

$$u_{tt} - c^2 \Delta u = 0, \quad (189)$$

where $\Delta u = u_{xx}u_{yy}$ in two spatial variables and $\Delta u = u_{xx}u_{yy}u_{zz}$ in three spatial variables.

As the name “wave” implies and as the solution formula for the initial-boundary value problem (IBVP) reveals, the fundamental solutions of the wave equation are oscillatory. Thus, the wave equation describes fundamentally different physical behaviors from those of the heat (or diffusion) equation, which works by tending to minimize gradients. It is important to bear in mind that the wave equation does not describe all linear waves. There is a different type of wave equations, called dispersive wave equations. Typical examples, in one space dimension, are the third order equation,

$$u_t + au_{xxx} = 0, \quad (190)$$

and the Schrödinger equation,

$$iu_t + au_{xx} = 0, \quad (191)$$

that also describe wave motion. The word “dispersive” reflects the fact that the speed of propagation of a wave of particular wavelength (or frequency) varies with the value of the wavelength (or frequency). This is in contrast to the wave equation that we are studying here, in which waves travel with the constant speed $c$, as one sees in (182). The Maxwell equations describe electromagnetic (EM) waves in the vacuum and in material media at small amplitudes and can be reduced to the wave equation. At higher amplitudes, nonlinear behavior creeps in, due to the interaction of the wave with the material. Fiber-optical EM
transmission is characterized by perturbations of the Nonlinear Schrödinger equation,

\[ iu_t + \varepsilon |u|^2 u + u_{xx} = 0, \]  

(192)
in which the coefficient \( \varepsilon \) of the so-called Kerr nonlinearity is small, but crucial for the phenomenology. Water waves, except at very small amplitudes are also dispersive.

In spite of the behavioral differences, between the wave and heat equations, the procedure for solving the IBVP is the same. We first write the equation as

\[ u_{tt} + c^2 L u = 0. \]  

(193)

We let

\[ u(x,t) = b_1(t) \psi_1(x) + b_2(t) \psi_2(x) + b_3(t) \psi_3(x) + \cdots, \]  

(194)

As with the heat equation, we obtain an infinite number of uncoupled ODE for the \( b_n \):

\[ \frac{d^2}{dt^2} b_n + c^2 \lambda_n b_n = 0. \]  

(195)

The general solution of the equation for \( b_n \) is

\[ b_n = A_n \cos(c \sqrt{\lambda_n} t) + B_n \sin(c \sqrt{\lambda_n} t). \]  

(196)

The time evolution of each of the solution modes is oscillatory, in stark contrast to the behavior of the heat equation. The constants \( A_n \) and \( B_n \) are obtained from our knowledge of \( b_n(0) \) and \( \dot{b}_n(0) \) (a dot indicates a time derivative).
The Laplace equation: A boundary value problem (BVP)

The Laplace equation in three spatial dimensions is

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0.$$  \hfill (197)

When it includes a non-homogeneous term,

$$u_{xx} + u_{yy} + u_{zz} = -\rho(x, y, z),$$  \hfill (198)

it is referred to as the Poisson equation.

The property of the Laplacian operator of being the divergence of the gradient,

$$\Delta u = \nabla \cdot \nabla u = \text{div} \text{ grad } u.$$  \hfill (199)

gives the Laplace and Poisson equations prominence in physics. The divergence of the field at a point in space is the nett field flux per unit volume, emanating from an infinitesimally small piece of space surrounding the point (divergence theorem in the limit of zero volume). The flux per unit volume can often be identified with field source density, or the spatial density of the physical sources that are responsible for maintaining the field. Such are mass density for the Newtonian gravitational field or charge density for the electrostatic field. Then,

$$\text{div } \mathbf{F} = \rho(x, y, z),$$  \hfill (200)

where $\mathbf{F}(x, y, z)$ is the force field and $\rho(x, y, z)$ is its source density.

Furthermore, many force fields $\mathbf{F}(x, y, z)$, including the electrostatic and the Newtonian gravitational fields, are derived as the negative gradient of a scalar field $u(x, y, z)$ called potential,

$$\mathbf{F}(x, y, z) = -\nabla u.$$  \hfill (201)
Inserting (201) into (200) obtains the potential must satisfy the Poisson equation (198). It happens often that all field sources lie outside our region of interest in the field. The potential satisfies Laplace’s equation (197) in this region.

The Laplace and Poisson equations are often posed on a bounded domain, for example inside of a sphere, with a boundary condition all around the boundary. We solve the Laplace equation in two space dimensions on a semi-infinite strip. We let

\[ \Delta u = u_{xx} + u_{yy} = 0, \quad (202) \]

inside the semi-infinite strip of the \( x, y \) plane defined by \( 0 < x < L \) and \( y > 0 \). We impose one boundary condition along the half-line \( x = 0, \ y > 0 \) and one along the half-line \( x = L, \ y > 0 \). In order to be able to apply directly the techniques that we have developed, we require these boundary conditions to be homogeneous. We also impose a boundary condition, generally non-homogeneous, on the interval \( 0 < x < L \) on the real axis. Finally, we require that the solution to the Laplace equation be bounded in the strip, that is it must remain smaller in absolute value than some finite value. This prohibits the solution to grow without bound as \( y \to +\infty \); thus, it is essentially a boundary condition at \( y = +\infty \).

As in the case of the heat equation, we write the PDE as

\[ u_{yy} + \mathbb{L} u = 0, \quad \mathbb{L} = \frac{d^2}{dx^2}, \quad (203) \]

**Example 1: Dirichlet boundary conditions.** We choose zero Dirichlet boundary conditions at \( x = 0 \) and \( x = L, \) that is \( u(0, y) = 0 \) and \( u(L, y) = 0. \) We also choose the Dirichlet condition

\[ u(x, 0) = h(x) = b_1(0)\psi_1(x) + b_2(0)\psi_2(x) + b_3(0)\psi_3(x) + \cdots, \quad \psi_n(x) = \sin \frac{n\pi x}{L}, \quad (204) \]

on the interval \((0, L)\) on the \( x \) axis, where \( \psi_n(x) \) is the eigenfunction corresponding to the
eigenvalue $\lambda_n$ of the operator $\mathbb{L}$. The coefficients $h_n$ are calculated as usual from formula (145). We let

$$u(x, y) = b_1(y)\psi_1(x) + b_2(y)\psi_2(x) + b_3(y)\psi_3(x) + \cdots,$$

and insert it into (203). By the usual procedure, we obtain the following ODE for the $b_n$s.

$$\frac{d^2 b_n}{dy^2} - \lambda_n b_n = 0, \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \cdots,$$

with solution basis

$$e^{-\sqrt{\lambda_n}y} \text{ and } e^{\sqrt{\lambda_n}y}. \quad (207)$$

The solution $e^{\sqrt{\lambda_n}y}$ violates the boundedness condition as $y \to +\infty$ and is rejected. Thus,

$$u(x, y) = b_1(0)e^{-\frac{n\pi y}{L}} \sin \frac{\pi x}{L} + b_2(0)e^{-\frac{2n\pi y}{L}} \sin \frac{2\pi x}{L} + \cdots + b_n(0)e^{-\frac{n^2\pi y}{L}} \sin \frac{n\pi x}{L} + \cdots \quad (208)$$

The solution tends to zero exponentially, as $y \to +\infty$.

**Example 2: Neumann conditions and the existence of a nullspace.** We change the boundary conditions from Dirichlet to Neumann. So $u_x(0, y) = 0$, $u_x(L, y) = 0$ on the two half-lines of the boundary and $u_y(x, 0) = h(x)$ on the interval. In order to avoid jumps of the boundary condition, we require $h(0) = h(L) = 0$. The proper basis to use in representing the solution, is the eigenvectors of the operator $\mathbb{L}$ with Neumann conditions (see (135))

$$1, \cos \frac{\pi x}{L}, \cos \frac{2\pi x}{L}, \cdots, \cos \frac{n\pi x}{L}, \cdots,$$

with eigenvalues

$$0, \left(\frac{\pi x}{L}\right)^2, \left(\frac{2\pi x}{L}\right)^2, \cdots, \left(\frac{n\pi x}{L}\right)^2, \cdots,$$ (210)

respectively. So,

$$u(x, y) = b_0(y) + b_1(y)\cos \frac{\pi x}{L} + b_2(y)\cos \frac{2\pi x}{L} + \cdots + b_n(y)\cos \frac{n\pi x}{L} + \cdots \quad (211)$$
The Neumann boundary condition \( u_x = 0 \) is satisfied along the half-lines \( x = 0 \) and \( x = L \), as a result of our choice of basis. Inserting this representation into the PDE (with the second \( x \)-derivative replaced by \(-L\)), and balancing coefficients, we obtain the equations for the \( b_n(y) \),

\[
\frac{d^2b_n}{dy^2} - \lambda_n b_n = 0, \quad n = 0, 1, 2, 3, \ldots
\]  

(212)

For \( n = 1, 2, 3, \ldots \) and recalling the boundedness condition, the solution of this equation is,

\[
b_n(y) = b_n(0)e^{-\frac{n\pi y}{L}}, \quad n = 1, 2, 3, \ldots
\]  

(213)

For \( b_0(y) \), again recalling the boundedness condition, we have,

\[
b_0(y) = c, \quad \text{where } c \text{ is an arbitrary constant.}
\]  

(214)

We must now enforce the given boundary condition \( u_y(x, 0) \).

\[
u_y(x, 0) = 0 + b'_1(0)\cos\frac{\pi x}{L} + b'_2(0)\cos\frac{2\pi x}{L} + \cdots + b'_n(0)\cos\frac{n\pi x}{L} + \cdots
\]

\[
= 0 - \frac{\pi}{L}b_1(0)\cos\frac{\pi x}{L} - \frac{2\pi}{L}b_2(0)\cos\frac{2\pi x}{L} \cdots - \frac{n\pi}{L}b_n(0)\cos\frac{n\pi x}{L} - \cdots
\]  

(215)

We observe that

- For the problem to be solvable, the constraint on the boundary data

\[
\int_0^L u(x, 0)dx = 0,
\]  

(216)

must be satisfied (why?)

- If the constraint is satisfied, the problem has infinitely many solutions, due to the arbitrariness of the constant \( c \).

This happens because the Laplacian operator with Neumann boundary data has a zero eigenvalue, with the constant function as the corresponding eigenfunction.
Example 3: The Laplace equation inside a rectangle with Dirichlet boundary conditions on the four sides. Exercise.

Fourier series

A function \( f(x) \) is **periodic** if there is a scalar \( p \) such that for all \( x \),

\[
f(x + p) = f(x). \tag{217}
\]

The scalar \( p \) is called the **period** of the function. For example, the functions \( \cos x \) and \( \sin x \) are periodic with period \( p = 2\pi \); the function \( \tan x \) is periodic of period \( p = \pi \); the functions \( \cos \frac{nx}{L} \) and \( \sin \frac{nx}{L} \) are periodic of period \( p = 2L \). Consistent with the repeating nature of a periodic function, the inner product and the norm are defined in terms of integrals over a length equal to the period \( p \),

\[
(f, g) = \int_a^{a+p} f(x)g(x)dx, \quad \|f\| = \sqrt{\int_a^{a+p} |f(x)|^2dx}. \tag{218}
\]

The lower limit of integration \( a \) is arbitrary. We consider periodic functions of period \( p = 2L \) and we, typically, integrate over the interval \((-L, L)\).

A \( 2L \)-periodic function \( f(x) \) of finite norm \((\|f\| < \infty)\) can be written as a linear superposition of the eigenfunctions of the operator \(-\frac{d^2}{dx^2}\) acting on twice differentiable functions defined on the interval \((-L, L)\) and satisfying periodic boundary conditions. The set of these eigenfunctions

\[
\begin{align*}
\frac{1}{2}, & \quad \cos \frac{\pi x}{L}, & \quad \sin \frac{\pi x}{L}, & \quad \cos \frac{2\pi x}{L}, & \quad \sin \frac{2\pi x}{L}, & \quad \cos \frac{3\pi x}{L}, & \quad \sin \frac{3\pi x}{L}, & \cdots \\
& \quad e^{-value=0}, & \quad e^{-value=(\frac{\pi}{L})^2}, & \quad e^{-value=(\frac{2\pi}{L})^2}, & \quad e^{-value=(\frac{3\pi}{L})^2}, & \cdots
\end{align*}
\tag{219}
\]
have been derived in equations (137). This expansion of \( f(x) \) is

\[
f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left( a_m \cos \frac{m\pi x}{L} + b_m \sin \frac{m\pi x}{L} \right).
\] (220)

It is known as the Fourier series representation of the function \( f(x) \).

In order to calculate the coefficients in the Fourier series, known as the Fourier coefficients of the function \( f(x) \), one makes use of the orthogonality of the eigenfunctions with respect to each other, following the procedure of the derivation of the coefficients in equation (145). We obtain,

\[
a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, dx, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx.
\] (221)

The vector that has the Fourier coefficients as entries,

\[
(a_0, a_1, b_1, a_2, b_2, a_3, b_3, \ldots)
\] (222)

is the representation of the periodic function \( f(x) \) in the coordinate system given by the basis functions (219). Notice that the basis is orthogonal but not orthonormal,

\[
\left\| \frac{1}{2} \right\| = \sqrt{\frac{L}{2}}, \quad \left\| \cos \frac{n\pi x}{L} \right\| = \left\| \sin \frac{n\pi x}{L} \right\| = \sqrt{L} \quad (n = \pm 1, \pm 2, \ldots)
\] (223)

Exercise: Prove that the squared norm of the function \( f(x) \) is given by

\[
\|f\|^2 = L \frac{a_0^2}{2} + L \sum_{n=1}^{\infty} \left( a_n^2 + b_n^2 \right).
\] (224)

Exercise: What is the inner product of the \( 2L \)-periodic functions \( f(x) \) and \( \tilde{f}(x) \) in terms of their Fourier coefficients \([a_n \text{ s and } b_n \text{ s}] \) and \([\tilde{a}_n \text{ s and } \tilde{b}_n \text{ s}] \), respectively?

Reading from textbook: (a) The Fourier convergence theorem (b) The Gibbs phe-
nomenon.

**Alternative Fourier series representation.** Each nonzero eigenvalue is double. Rather than use the functions $\cos \frac{n\pi x}{L}$ and $\sin \frac{n\pi x}{L}$ as basis of the eigenspace corresponding to this eigenvalue, we may use

$$e^{\frac{in\pi x}{L}} \quad \text{and} \quad e^{-\frac{in\pi x}{L}}.$$  \hfill (225)

The following orthogonality relation applies (the complex conjugate of $e^{ia}$ is $e^{-ia}$ for real $a$),

$$\langle e^{\frac{in\pi x}{L}}, e^{\frac{im\pi x}{L}} \rangle = \int_{-L}^{L} e^{\frac{in\pi x}{L}} e^{-\frac{im\pi x}{L}} \, dx = \int_{-L}^{L} e^{i(n-m)\frac{\pi x}{L}} \, dx = \begin{cases} 0 & \text{if } m \neq n \\ 2L & \text{if } m = n. \end{cases}$$  \hfill (226)

In particular, when $n \neq 0$, we have $\langle e^{\frac{in\pi x}{L}}, e^{i(-n\pi x/L)} \rangle = 0$. The basis consisting of

$$e^{\frac{in\pi x}{L}}, \quad \text{where } n \text{ runs over all integers from } -\infty \text{ to } +\infty$$  \hfill (227)

is an orthogonal basis. The basis is still not orthonormal,

$$\left\| e^{\frac{in\pi x}{L}} \right\| = \sqrt{2L},$$  \hfill (228)

One may of course **normalize** the basis to

$$\phi_n(x) = \frac{1}{\sqrt{2L}} e^{\frac{in\pi x}{L}}, \quad \|\phi_n\| = 1.$$  \hfill (229)

The expansion of the function $f(x)$ in the basis (227) is

$$f(x) = \sum_{m=-\infty}^{\infty} c_m e^{\frac{im\pi x}{L}}.$$  \hfill (230)

In order to calculate the coefficient $c_n$, we take the inner product of both sides of (230) with
the basis function $e^{inx/L}$. We obtain

$$\int_{-L}^{L} f(x)e^{-inx/L} \, dx = 2Lc_n. \quad (231)$$

Thus,

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x)e^{-inx/L} \, dx. \quad (232)$$

The representation of the periodic function $f(x)$ as a vector with infinitely many entries is

$$c = (\cdots, c_{-3}, c_{-2}, c_{-1}, c_0, c_1, c_2, c_3, \cdots). \quad (233)$$

**Exercise:** What is the inner product of two functions in this representation?

**The Fourier transform**

We consider the eigenvalue problem for the operator $\mathbb{L} = -\frac{d^2}{dx^2}$ on the whole real $x$ line, acting on twice differentiable $L(-\infty, +\infty)$ functions,

$$\mathbb{L}\psi = \lambda\psi. \quad (234)$$

We obtain,

"eigenfunctions": $e^{ikx}$,  "eigenvalues": $\lambda = k^2$, $k \in \mathbb{R}$. \quad (235)

When $k$ has a nonzero imaginary part, the function $e^{ikx}$ displays exponential growth and is inadmissible. Even with the admitted real values of $k$, the "eigenfunctions" $e^{ikx}$ are not in the space $L(-\infty, +\infty)$. Nevertheless, we want to use the "eigenfunctions" $e^{ikx}$ as an improper kind of basis of $L(-\infty, +\infty)$. We are forced to take linear superpositions of the basis functions using integration instead of summation. This reflects the fact that $k$ is not a discrete parameter, but takes values over a continuum. We want to represent a function
f(x) in \( L(-\infty, +\infty) \) as
\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx} dk,
\]
(236)
where \( \frac{\hat{f}(k)}{2\pi} \) plays the role of the coefficient in our earlier series representations and \( k \) is a continuous analogue of the index of summation in series eigenfunction expansions.

As we did with proper eigenfunctions, we will establish an “orthogonality” relation between \( e^{ikx} \) and \( e^{ilx} \) when \( k \neq l \), ignoring momentarily that the functions involved are not in \( L^2 \). To prove is the relation
\[
\langle e^{ikx}, e^{ilx} \rangle = \int_{-\infty}^{\infty} e^{ikx}e^{-ilx} dx = 2\pi \delta(k-l),
\]
(237)
where \( \delta \) is the Dirac delta function. Recalling that the complex conjugate of \( e^{ilx} \) is \( e^{-ilx} \) and letting \( k - l = a \), we can rewrite this relation as
\[
\int_{-\infty}^{\infty} e^{iax} dx = 2\pi \delta(a).
\]
(238)
The integral on the left is not a classical improper integral. If we integrate it in a large interval \([-N, +N]\), we find the value \( \frac{2\sin(aN)}{a} \). This oscillates as \( N \to \infty \), there is no limit. For such oscillatory integrals, instead of the sharp cut-off at \( \pm N \), we make a smooth cut-off by multiplying the integrand by the factor \( e^{-\varepsilon x^2} \). When \( \varepsilon > 0 \) is small, the effect of the factor is felt only at high values of \( x \) providing superexponential decay as \( x \to \infty \). As the smooth cut-off is removed in the limit \( \varepsilon \to 0 \), we obtain the right hand side of equation (238). The following string of equations presents the procedure. Explanations follow.
\[
\int_{-\infty}^{\infty} e^{iax} dx = \lim_{\varepsilon \searrow 0} \int_{-\infty}^{\infty} e^{iax - \varepsilon x^2} dx = 2\pi \lim_{\varepsilon \searrow 0} \left( \frac{e^{-\frac{a^2}{4\varepsilon}}}{\sqrt{4\pi\varepsilon}} \right) = 2\pi \delta(a).
\]
(239)
We justify the last equality first. The quantity in parenthesis is the normal probability distribution; its graph is the well-known bell-shaped gaussian curve. The area under the graph equals 1 (probability curve). As \( \varepsilon \searrow 0 \), the distribution becomes tall, with its mass
concentrated near $a = 0$. It tends to the Dirac delta function. We now calculate the integral

$$\int_{-\infty}^{\infty} e^{iax-\varepsilon x^2} dx = \int_{-\infty}^{\infty} e^{-\varepsilon x^2} \cos(ax) dx + i \int_{-\infty}^{\infty} e^{-\varepsilon x^2} \sin(ax) dx. \tag{240}$$

The integral that involves the sine equals zero (odd integrand). The integral that involves the cosine is independent of $a$, because its derivative with respect to $a$ is zero.

$$\frac{d}{da} \int_{-\infty}^{\infty} e^{-\varepsilon x^2} \cos(ax) dx = -a \int_{-\infty}^{\infty} e^{-\varepsilon x^2} \sin(ax) dx = 0 \tag{241}$$

Thus, we can let $a = 0$ in (240) without altering the value of the integrals. The integral we need to compute to complete the calculation simplifies to

$$J = \int_{-\infty}^{\infty} e^{-\varepsilon x^2} dx. \tag{242}$$

There is a famous trick for evaluating this important integral (its the gaussian). Calculate $J^2$ and use polar coordinates.

$$J^2 = \left( \int_{-\infty}^{\infty} e^{-\varepsilon x^2} dx \right) \left( \int_{-\infty}^{\infty} e^{-\varepsilon y^2} dy \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\varepsilon (x^2+y^2)} dxdy = 2\pi \int_{0}^{\infty} e^{-\varepsilon r^2} r dr = \frac{\pi}{\varepsilon}. \tag{243}$$

The result for $J$ is

$$\int_{-\infty}^{\infty} e^{-\varepsilon x^2} dx = \sqrt{\frac{\pi}{\varepsilon}}. \tag{244}$$

This completes the proof of relation (238).

We now apply orthogonality to calculate $\hat{f}(k)$ in the representation (236).

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(l)e^{ilx} dl \tag{245}$$

$$(f(x), e^{ikx}) = \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} \hat{f}(l)e^{ilx} dl, e^{ikx} \right) \tag{246}$$
The next step involves a change in the order of integration.

\[(f(x), e^{ikx}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(l)(e^{ilx}, e^{ikx})dl\]  

(247)

We use the orthogonality relation (237)

\[(f(x), e^{ikx}) = \int_{-\infty}^{\infty} \hat{f}(l)\delta(l - k)dl\]  

(248)

\[\int_{-\infty}^{\infty} \hat{f}(x)e^{-ilx} = \hat{f}(l),\]  

(249)

The pair of relations

\[\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ilx}, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk\]  

(250)

are known as the **Fourier Transform** and the **Inverse Fourier Transform**, respectively.

Some important properties of the Fourier transform (FT).

- FT is a linear transformation
- FT: \( \frac{d}{dx} \) to \( ik \).
- Convolution \( f * g = \int_{-\infty}^{\infty} f(x - y)g(y)dy \) to \( \hat{f}(k)\hat{g}(k) \)
- for \( f \) and \( g \) in \( L^2 \), \( (f, g) = \frac{1}{2\pi} (\hat{f}, \hat{g}) \)
- for \( f \) in \( L^2 \), \( \|f\|^2 = \frac{1}{2\pi} \|\hat{f}\|^2 \) (equal energy in space and in frequency)

**PDE Application of Fourier Transform: The Schrödinger Equation (IVP).**

Solve the initial value problem (IVP)

\[iu_t + u_{xxx} = 0, \quad x \in \mathbb{R}, \quad t > 0,\]  

(251)
with initial condition

\[ u(x, 0) = f(x). \quad (252) \]

**Solution:** Take the Fourier transform of the PDE

\[ \hat{u}_t + \hat{u}_{xx} = 0 \quad (253) \]

\[ i\hat{u}_t + (ik)^2 \hat{u} = 0, \quad \hat{u}_t + ik^2 \hat{u} = 0, \quad (254) \]

\[ \hat{u}(k, t) = c(k)e^{-ik^2t} \quad (255) \]

Determine the constant \(c(k)\) using the initial condition: \( \hat{u}(k, 0) = c(k) \), thus, \( c(k) = \hat{f}(k) \).

The Fourier transform of the solution of the IVP is,

\[ \hat{u}(k, t) = \hat{f}(k)e^{-ik^2t} \quad (256) \]

Apply the inverse Fourier transform in order to obtain the solution.

\[ u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)e^{ikx-ik^2t} \, dk. \quad (257) \]

**PDE Application of Fourier Series:** The wave equation (IBVP).

Solve the initial-boundary value problem (IBVP)

\[ u_{tt} - c^2 u_{xx} = 0, \quad -L < x < L, \quad t > 0, \quad (258) \]

with periodic boundary conditions and initial conditions

\[ u(x, 0) = e^{\frac{2\pi ix}{L}}, \quad u_t(x, 0) = 0. \quad (259) \]
Solution: The PDE can be written as

\[ u_{tt} + c^2 L u = 0, \]  

(260)

where \( L = -\frac{d^2}{dx^2} \) with periodic boundary conditions. There is no forcing on the PDE (no non-homogeneous term) and there is no non-homogeneous term in the boundary conditions. The initial conditions involve only a single eigenfunction of the operator \( L \), namely the eigenfunction \( e^{\frac{2\pi i x}{L}} \), and this constitutes the only nonzero input to the system. Our procedure below does not rely on the nature of the particular eigenfunction. To emphasize the generality of the argument, we call the eigenfunction and the corresponding eigenvalue by their generic names, \( \psi(x) \) and \( \lambda \), respectively. The central point is that, as time changes from its initial value, the solution will still involve only the eigenfunction \( \psi(x) \). The footprint of time appears only as a scalar multiplier of the eigenfunction. Solutions, that involve several (possibly infinitely many) input eigenfunctions can be handled separately for each input eigenfunction and summed together at the end by invoking the principle of linear superposition. In symbols, the solution of the above problem has form

\[ u(x, t) = b(t)\psi(x). \]  

(261)

The truth of this is made obvious, by calculating the factor \( b(t) \) directly. One inserts (261) into equation (260). Recalling that

\[ L \psi = \lambda \psi, \]  

(262)

and pulling \( \psi \) out as a common factor, one obtains

\[ \{\ddot{b}(t) + c^2 \lambda b(t)\} \psi(x) = 0. \]  

(263)
This is satisfied if \( b \) satisfies the ODE

\[
\ddot{b}(t) + c^2 \lambda b(t) = 0. \tag{264}
\]

The general solution of this ODE is

\[
b(t) = c_1 e^{ic\sqrt{\lambda}t} + c_2 e^{-ic\sqrt{\lambda}t}. \tag{265}
\]

The constants \( c_1 \) and \( c_2 \) are calculated by utilizing the initial data. The expressions

\[
e^{ic\sqrt{\lambda}t} \psi(x) \quad \text{and} \quad e^{-ic\sqrt{\lambda}t} \psi(x), \tag{266}
\]

are space-periodic solutions of the wave equation with periodic conditions of period \( 2L \).

Typically in the physics and engineering literature, the solutions of a homogeneous linear system, like the ones in (266), are referred to as the “modes” of the system. All that the system can do, when left undisturbed, is given by the linear superpositions of the modes. The coefficients of a superposition may be determined from knowledge of the initial condition of the system. The calculation for the solution of a homogeneous linear selfadjoint IBVP proceeds as follows:

- Choose the eigenfunction basis that is consistent with the (homogeneous) boundary conditions.
- Calculate the coefficients of the eigenfunction expansion(s) of the initial data.
- For each participating eigenfunction, derive and solve the ODE that determines the time-evolution of the coefficient \( b(t) \). This determines the solution mode corresponding to the eigenfunction. The results of the previous item provide the initial data for the ODE in question.
- Sum the mode contributions from all participating eigenfunctions. The solution is a
(generally infinite) linear superposition of modes.

Let us now complete the solution of the problem at hand. The initial conditions for \( b(t) \) are

\[
b(0) = 1, \quad \dot{b}(0) = 0,
\]

which implies \( c_1 + c_2 = 1 \) and \( c_1 - c_2 = 0 \). Thus, \( c_1 = c_2 = \frac{1}{2} \). Recalling that

\[
\lambda = \left( \frac{2\pi}{L} \right)^2, \quad \psi(x) = e^{\frac{2\pi x}{L}},
\]

we obtain

\[
b(t) = \frac{1}{2} \left( e^{\frac{2\pi ct}{L}} + e^{-\frac{2\pi ct}{L}} \right) = \cos \frac{2\pi ct}{L}, \quad u(x,t) = \cos \left( \frac{2\pi ct}{L} \right) e^{\frac{2\pi x}{L}}
\]

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Writing the cosine in terms of exponentials, we obtain

\[
u(x,t) = \frac{1}{2} \left( e^{\frac{2\pi i(x-ct)}{L}} + e^{\frac{2\pi i(x+ct)}{L}} \right).
\]

This is the sum of two exponential waves traveling in opposite directions. Sometimes, the physical quantity of interest is the real part of \( u \). After a simple calculation,

\[
\Re u(x,t) = \cos \frac{2\pi ct}{L} \cos \frac{2\pi x}{L}.
\]

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This is a standing wave.

Disturbing or forcing a system, like the one studied above, is done by adding non-homogeneous terms to the PDE. Non-homogeneous terms in the boundary conditions can be eliminated at the cost of extra non-homogeneous terms in the PDE. The approach to solving the IBVP problem, remains essentially the same as above, except that it includes the handling of the nonhomogeneity.

- Choose the eigenfunction basis that is consistent with the (homogeneous) boundary
conditions.

- Calculate the coefficients of the eigenfunction expansion of the initial data.

- Calculate the coefficients of the eigenfunction expansion of the non-homogeneous term.

- An eigenfunction participates if it has a nonzero coefficient in any of the above expansions. For each of these eigenfunctions, derive and solve the ODE that determines the time-evolution of the coefficient $b(t)$. The eigenfunctions that contribute to the non-homogeneous term will have a non-homogeneous contribution to the corresponding ODE for $b(t)$.

- Sum the contributions to the solution from all eigenfunctions.

**Hint:** In the problems below, work with one eigenfunction at a time.

**Exercise:** Solve the problem of the example, if the initial condition $u_t = 0$ is replaced by $u_t = 2e^{\frac{3\pi}{L}x}$.

**Exercise:** What is the solution if the initial conditions are $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$?

**Exercise:** What is the solution if there is a forcing term $h(x, t)$ in the PDE?

**Exercise:** Add a non-homogeneous term to the PDE that does not grow in time and produces a solution that grows in time. Take both initial conditions to be equal to zero.

**Exercise:** Do you agree with the statement that the solution corresponding to an eigenfunction behaves like a classical harmonic oscillator? Discuss.

**PDE Application:** The Laplace and Poisson equations in a strip and in a rectangle.