Cellular resolutions of sums of multiplier ideals

Ezra Miller
(joint work with Shin-Yao Jow)

Let X be a smooth complex algebraic variety and let $\mathfrak{a} \subseteq \mathcal{O}_X$ be an ideal sheaf. Applications in algebraic geometry of the multiplier ideal sheaves

$$\mathcal{J}(\mathfrak{a}^\alpha) = \mathcal{J}(X, \mathfrak{a}^\alpha) \subseteq \mathcal{O}_X$$

for real numbers $\alpha > 0$ have led to investigations of their behavior with respect to natural algebraic operations. For example, Demailly, Ein, and Lazarsfeld [DEL00] proved that given two ideal sheaves $\mathfrak{a}_1$ and $\mathfrak{a}_2$, one has

$$\mathcal{J}((\mathfrak{a}_1\mathfrak{a}_2)^\alpha) \subseteq \mathcal{J}(\mathfrak{a}_1^\alpha)\mathcal{J}(\mathfrak{a}_2^\alpha).$$

For the subtler case of sums, on the other hand, Mustaţă [Mus02] showed that

$$\mathcal{J}((\mathfrak{a}_1 + \mathfrak{a}_2)^\alpha) \subseteq \sum_{0 \le t \le \alpha} \mathcal{J}(\mathfrak{a}_1^{\alpha-t})\mathcal{J}(\mathfrak{a}_2^t),$$

and Takagi [Tak05] later refined this to

$$\mathcal{J}((\mathfrak{a}_1 + \mathfrak{a}_2)^\alpha) = \sum_{0 \le t \le \alpha} \mathcal{J}(\mathfrak{a}_1^{\alpha-t}\mathfrak{a}_2^t), \tag{1}$$

where he proved it more generally when X is $\mathbb{Q}$-Gorenstein.

Takagi used characteristic p methods to deduce (1), which makes his work distinctly different from [DEL00] and [Mus02], where the arguments proceed by geometric techniques such as log resolutions and sheaf cohomology. Our purpose is to show that such geometric techniques, combined with combinatorial methods from topology and commutative algebra, can recover Takagi's equality (1) and generalize it. As a consequence, we derive a new proof of Howald's formula for multiplier ideals of monomial ideals [How01], and demonstrate how it can be reformulated to hold for all ideals. Our main results center around the following.

Theorem 1. *Fix nonzero ideal sheaves $\mathfrak{a}_1, \ldots, \mathfrak{a}_r, \mathfrak{b}$ on a $\mathbb{Q}$-Gorenstein complex variety X. For any real $\alpha, \beta > 0$, there is a resolution $0 \to \mathcal{J}_r \to \cdots \to \mathcal{J}_0 \to 0$ of the multiplier ideal $\mathcal{J}((\mathfrak{a}_1 + \cdots + \mathfrak{a}_r)^\alpha \mathfrak{b}^\beta)$ by sheaves $\mathcal{J}_i$ that are finite direct sums of multiplier ideals of the form $\mathcal{J}(\mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r} \mathfrak{b}^\beta)$ for various nonnegative $\lambda \in \mathbb{R}^r$ with $\sum_{i=1}^r \lambda_i = \alpha$. Every distinct ideal sheaf of that form appears as a summand of $\mathcal{J}_0$.*

Part of the final claim of Theorem 1 is that there are only finitely many distinct multiplier ideals of the form $\mathcal{J}(X, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r} \mathfrak{b}^\beta)$ for $\lambda_1 + \cdots + \lambda_r = \alpha$. In particular, the surjection $\mathcal{J}_0 \twoheadrightarrow \mathcal{J}(X, (\mathfrak{a}_1 + \cdots + \mathfrak{a}_r)^\alpha \mathfrak{b}^\beta)$ in our resolution implies the following (finite) summation formula.

Corollary 2. $\mathcal{J}(X, (\mathfrak{a}_1 + \cdots + \mathfrak{a}_r)^\alpha \mathfrak{b}^\beta) = \displaystyle\sum_{\lambda_1 + \cdots + \lambda_r = \alpha} \mathcal{J}(X, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r} \mathfrak{b}^\beta).$

Corollary 2 reduces the calculation of the multiplier ideals of arbitrary polynomial ideals to those of principal ideals. In the special case of a monomial ideal $\mathfrak{a} = \langle \mathbf{x}^{\gamma_1}, \ldots, \mathbf{x}^{\gamma_r} \rangle$, generated by the monomials in the polynomial ring $\mathbb{C}[x_1, \ldots, x_d]$ with exponent vectors $\gamma_1, \ldots, \gamma_r \in \mathbb{N}^d$, the summation formula becomes particularly explicit. For a subset $\Gamma \subseteq \mathbb{R}^d$, let $\operatorname{conv}\Gamma$ denote its convex hull. By the *integer part* of a vector $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{R}^d$, we mean the vector $(\lfloor \nu_1 \rfloor, \ldots, \lfloor \nu_d \rfloor) \in \mathbb{Z}^d$ whose entries are the greatest integers less than or equal to the coordinates of ν.

Corollary 3 (Howald)**.** *If $\mathfrak{a} = \langle \mathbf{x}^{\gamma_1}, \ldots, \mathbf{x}^{\gamma_r} \rangle$ is a monomial ideal in $\mathbb{C}[x_1, \ldots, x_d]$, then $\mathcal{J}(\mathfrak{a}^\alpha)$ is generated by the monomials in $\mathbb{C}[x_1, \ldots, x_d]$ whose exponent vectors are the integer parts of the vectors in* $\operatorname{conv}\{\alpha \cdot \gamma_1, \ldots, \alpha \cdot \gamma_r\} \subseteq \mathbb{R}^d$.

Proof. Using the log resolution definition of multiplier ideals (not reproduced here; see [Laz04]) and Corollary 2 with $\mathfrak{a}_j = \langle \mathbf{x}^{\gamma_j} \rangle$, it suffices to note that the divisor of a monomial has simple normal crossings, so no log resolution is necessary. $\square$

It is easy to check that for $\alpha = 1$, the vectors in the conclusion of Corollary 3 are precisely those from Howald's result [How01], namely those $\gamma \in \mathbb{N}^d$ such that $\gamma + (1, \ldots, 1)$ lies interior to the convex hull of all exponents of monomials in $\mathfrak{a}$.

Our approach to Theorem 1 is to construct a specific resolution satisfying the hypotheses, including the part about $\mathcal{J}_0$. The resolution we construct is *cellular*, in a sense generalizing the manner in which resolutions of monomial ideals can be cellular [BS98]; see [MS05, Chapter 4] for an introduction. In general, a complex in any abelian category could be called *cellular* if each homological degree is a direct sum indexed by the faces of a CW-complex, and the boundary maps are determined in a natural way from those of the CW-complex. An elementary way to phrase this in the present context is as follows.

Theorem 4. *Resume the notation from Theorem 1. There is a triangulation Δ of the simplex $\{\lambda \in \mathbb{R}^r \mid \sum_{i=1}^r \lambda_i = \alpha\}$ such that we can take*

$$\mathcal{J}_i = \bigoplus_{\sigma \in \Delta_i} \mathcal{J}_\sigma$$

to be a direct sum indexed by the set Δ_i of i-dimensional faces $\sigma \in \Delta$, and the differential of $\mathcal{J}_\bullet$ is induced by natural maps between ideal sheaves, using the signs from the boundary maps of Δ. If $\lambda \in \Delta_0$ is a vertex then $\mathcal{J}_\lambda = \mathcal{J}(X, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r} \mathfrak{b}^\beta)$.

Comparing the final sentences of Theorems 1 and 4, a key point is that every possible multiplier ideal of the form $\mathcal{J}(X, \mathfrak{a}_1^{\lambda_1} \cdots \mathfrak{a}_r^{\lambda_r} \mathfrak{b}^\beta)$ occurs at some vertex $\lambda \in \Delta_0$. Writing down what it means for two such multiplier ideals to coincide, this stipulation provides strong hints as to the choice of triangulation.

The proof of exactness for the cellular resolution in Theorem 4 proceeds by lifting the problem to an appropriate log resolution $X' \to X$. Over X', we resolve the lifted ideal sheaf by a complex (of locally principal ideal sheaves in $\mathcal{O}_{X'}$) that is, analytically locally at each point of X', a cellular free complex over a polynomial ring. This cellular free complex turns out to be a cellular free resolution of an appropriate monomial ideal. Having cellularly resolved the lifted sheaf over X', the desired cellular resolution over $\mathcal{O}_X$ is obtained by pushing forward to X, using local vanishing for multiplier ideals [Laz04, Theorem 9.4.4]. Thus Theorems 1 and 4 constitute a certain global version of cellular free resolutions of monomial ideals.

The acyclicity of the monomial cellular free resolution reduces to a simplicial homology vanishing statement for simplicial complexes that are obtained by deleting boundary faces from certain contractible manifolds-with-boundary. We deduce this vanishing statement from the following more general result, which is of independent interest. Its hypothesis is satisfied by the barycentric subdivision of any polyhedral homology-manifold-with-boundary. (To *delete a simplex* σ from a simplicial complex M means to remove from M every simplex containing σ.)

Proposition 5. *Fix a simplicial complex M whose geometric realization $|M|$ is a homology-manifold with boundary ∂M. Assume M satisfies the following condition:*

$$\text{if } \sigma \text{ is a face of } M\text{, then } \sigma \cap |\partial M| \text{ is a face of } M.$$

Then deleting any collection of boundary simplices from M results in a simplicial subcomplex whose (reduced) homology is canonically isomorphic to that of M.

References

[BS98] Dave Bayer and Bernd Sturmfels, *Cellular resolutions of monomial modules*, J. Reine Angew. Math. **502** (1998), 123–140.

[DEL00] Jean-Pierre Demailly, Lawrence Ein and Robert Lazarsfeld, *A subadditivity property of multiplier ideals*, Michigan Math. J. **48** (2000), 137–156.

[How01] Jason Howald, *Multiplier ideals of monomial ideals*, Trans. Amer. Math. Soc. **353** (2001), no. 7, 2665-2671 (electronic).

[Laz04] Robert Lazarsfeld, *Positivity in Algebraic Geometry I–II*, Ergeb. Math. Grenzgeb., vols. **48**–**49**, Berlin: Springer, 2004.

[MS05] Ezra Miller and Bernd Sturmfels, *Combinatorial Commutative Algebra*, Graduate Texts in Mathematics **227**, New York: Springer, 2005.

[Mus02] Mircea Mustaţă, *The multiplier ideals of a sum of ideals*, Trans. Amer. Math. Soc. **354** (2002), no. 1, 205–217.

[Tak05] Shunsuke Takagi, *Formulas for multiplier ideals on singular varieties*, Amer. J. Math. **128** (2006), 1345–1362. `arXiv:math.AG/0410612`