BINOMIAL D-MODULES

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The authors dedicate this work to the memory of Karin Gatermann, friend and colleague

ABSTRACT. We study quotients of the Weyl algebra by left ideals whose generators consist of an arbitrary $\mathbb{Z}^d$-graded binomial ideal I in $\mathbb{C}[\partial_1, \ldots, \partial_n]$ along with Euler operators defined by the grading and a parameter $\beta \in \mathbb{C}^d$. We determine the parameters β for which these D-modules (i) are holonomic (equivalently, regular holonomic, when I is standard-graded); (ii) decompose as direct sums indexed by the primary components of I; and (iii) have holonomic rank greater than the rank for generic β. In each of these three cases, the parameters in question are precisely those outside of a certain explicitly described affine subspace arrangement in $\mathbb{C}^d$. In the special case of Horn hypergeometric D-modules, when I is a lattice basis ideal, we furthermore compute the generic holonomic rank combinatorially and write down a basis of solutions in terms of associated A-hypergeometric functions. This study relies fundamentally on the explicit lattice point description of the primary components of an arbitrary binomial ideal in characteristic zero, which we derive in our companion article [DMM08].

1. INTRODUCTION

Hypergeometric systems of Horn type, which have been studied since the late nineteenth century, provide the most direct multivariate generalization of the Gauss hypergeometric equation.

Definition 1.1. For a matrix $B \in \mathbb{Z}^{n\times m}$ with rows $b_1, \ldots, b_n$, and a vector $c = (c_1, \ldots, c_n)$ in $\mathbb{C}^n$, define elements of the Weyl algebra D_m in variables $z_1, \ldots, z_m\ \partial_{z_1}, \ldots, \partial_{z_m}$:

$$q_k(\theta_z) = \prod_{b_{jk}>0} \prod_{\ell=0}^{b_{jk}-1} (b_j \cdot \theta_z + c_j - \ell) \quad \text{and} \quad p_k(\theta_z) = \prod_{b_{jk}<0} \prod_{\ell=0}^{|b_{jk}|-1} (b_j \cdot \theta_z + c_j - \ell),$$

where $\theta_z = (\theta_{z_1}, \ldots, \theta_{z_m})$, $\theta_{z_k} = z_k \partial_{z_k}$ $(1 \le k \le m)$. The *classical Horn system with parameter* c is the left D_m-ideal $\mathrm{Horn}(B, c) = \langle q_k(\theta_z) - z_k p_k(\theta_z) \mid k = 1, \ldots, m\rangle$.

Although these objects appear in many places, basic questions remained unanswered, such as finding a dimension formula for the solutions, determining when this dimension is minimal, when it is finite, and explaining why certain solutions have smaller than expected support (see, for instance, [Erd50]). To study these problems, we use a change of variables, developed in [GGZ87, GKZ89]. The classical Horn system will be replaced by a binomial Horn system as in Definition 1.5 below.

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We adopt the following notation throughout.

Convention 1.2. $B = (b_{jk}) \in \mathbb{Z}^{n\times m}$ denotes a matrix of full rank $m \leq n$. We assume that B is *mixed*: any nonzero integer vector in the column span of B has two entries with different signs. We write $b_1, \dots, b_n$ for the rows of B.

Convention 1.3. $A = (a_{ij}) \in \mathbb{Z}^{d\times n}$ denotes a matrix whose columns $a_1, \dots, a_n$ $\mathbb{Z}$-span $\mathbb{Z}^d$. We assume that $a_1, \dots, a_n$ all lie in a single open linear half-space of $\mathbb{R}^d$; equivalently, the cone generated by the columns of A is pointed (contains no lines), and $a_i \neq 0$ for all i.

Convention 1.4. For B is as in Convention 1.2, set $d = n - m$. The mixedness of B implies that we can choose a (pointed) matrix A as in Convention 1.3 satisfying $AB = 0$. Note that we do allow $d = 0$, in which case A is the empty matrix.

Definition 1.5. Let B and A as in Convention 1.4. Consider the *lattice basis ideal* corresponding to B, that is, the ideal $I(B)$ in $\mathbb{C}[\partial]$ generated by the binomials

$$\prod_{b_{jk}>0} \partial_j^{b_{jk}} - \prod_{b_{jk}<0} \partial_j^{-b_{jk}} \quad \text{for} \quad 1 \leq k \leq m.$$

The *binomial Horn system with parameter* β is the left ideal

$$H(B,\beta) = I(B) + \langle \textstyle\sum_{j=1}^n a_{ij} x_j \partial_j - \beta_i \mid i = 1, \dots, n-m \rangle$$

in the Weyl algebra $D = D_n$ with variables $x_1, \dots, x_n$ and derivations $\partial_1, \dots, \partial_n$.

The classical-to-binomial transformation proceeds via the surjection

$$\begin{array}{ccc} (\mathbb{C}^*)^n & \to & (\mathbb{C}^*)^m \\ (x_1, \dots, x_n) & \mapsto & x^B = (\prod_{j=1}^n x_j^{b_{j1}}, \dots, \prod_{j=1}^n x_j^{b_{jm}}), \end{array} \tag{1.1}$$

where $\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$. A solution $f(z_1, \dots, z_m)$ of the classical Horn system $\mathrm{Horn}(B, c)$ gives rise to a solution $x^c f(x^B)$ of the binomial Horn system $H(B, Ac)$. When the columns of B are a basis of the integer kernel of A, this map defines a vector space isomorphism between the (local) solution spaces. The transformation $f(z) \mapsto x^c f(x^B)$ takes classical series solutions supported on $\mathbb{N}^m$ to Puiseux series solutions supported on the translate $c + \ker(A) \subseteq \mathbb{C}^n$ of the kernel of A in $\mathbb{Z}^n$. (Note that $\ker(A)$ contains the lattice $\mathbb{Z}B$ spanned by the columns of B as a finite index subgroup.) More precisely, the differential equations $\sum a_{ij} x_j \partial_j - \beta_i$, which geometrically impose torus-equivariance infinitesimally under the action of (the Lie algebra of) $\ker((\mathbb{C}^*)^n \to (\mathbb{C}^*)^m)$, result in series supported on $c + \ker(A)$, while the binomials in the lattice basis ideal $I(B) \subseteq H(B, Ac)$ impose hypergeometric constraints on the coefficients.

The central objects of study in this article are *binomial D-modules* (Definition 1.6), which generalize the above binomial Horn systems.

A matrix $A \in \mathbb{Z}^{d\times n}$ induces a $\mathbb{Z}^d$-grading of the polynomial ring $\mathbb{C}[\partial_1, \dots, \partial_n] = \mathbb{C}[\partial]$, which we call the A-*grading*, by setting $\deg(\partial_i) = -a_i$. An ideal of $\mathbb{C}[\partial]$ is A-*graded* if it is generated by elements that are homogeneous for the A-grading. For example, a *binomial ideal* is generated by *binomials* $\partial^u - \lambda\partial^v$, where $u, v \in \mathbb{Z}^n$ are column vectors and $\lambda \in \mathbb{C}$; such an ideal is A-graded precisely when it is generated by binomials $\partial^u - \lambda\partial^v$ each of which satisfies either $Au = Av$ or $\lambda = 0$ (in particular, monomials are allowed as generators of binomial ideals). The hypotheses on A made in Convention 1.3 mean that the A-grading is a *positive* $\mathbb{Z}^d$*-grading* [MS05, Chapter 8]. The Weyl algebra $D = D_n$ of linear partial differential operators, written with the variables x and ∂, is also naturally A-graded by additionally setting $\deg(x_i) = a_i$. Consequently, the *Euler operators* in our next definition are A-homogeneous of degree 0.

Definition 1.6. For each $i \in \{1, \dots, d\}$, the i^{th} *Euler operator* is

$$E_i = a_{i1}x_1\partial_1 + \cdots + a_{in}x_n\partial_n.$$

Given a vector $\beta \in \mathbb{C}^d$, we write $E - \beta$ for the sequence $E_1 - \beta_1, \dots, E_d - \beta_d$.

For an A-graded binomial ideal $I \subseteq \mathbb{C}[\partial]$, we denote by $H_A(I, \beta)$ the left ideal $I + \langle E - \beta \rangle$ in the Weyl algebra D. The *binomial D-module* associated to I is $D/H_A(I, \beta)$.

The fundamental examples of binomial D-modules, and the ones which our definition most directly generalizes, are the *A-hypergeometric systems* (or *GKZ hypergeometric systems*) of Gelfand, Graev, Kapranov, and Zelevinsky [GGZ87, GKZ89]. Given A as in Convention 1.3, these are the left D-ideals $H_A(I_A, \beta)$, also denoted by $H_A(\beta)$, where

$$I_A = \langle \partial^u - \partial^v : Au = Av \rangle \subseteq \mathbb{C}[\partial_1, \dots, \partial_n] \tag{1.2}$$

is the *toric ideal* for the matrix A. The systems $H_A(\beta)$ have many applications; for example, they arise naturally in the moduli theory of Calabi-Yau complete intersections in toric varieties, and (therefore) they play an important role in applications of mirror symmetry in mathematical physics [BvS95, Ho99, Hos06, HLY96]. We recall that the ideal I_A is a prime A-graded binomial ideal, and the quotient ring $\mathbb{C}[\partial]/I_A$ is the semigroup ring for the affine semigroup $\mathbb{N}A$ generated by the columns of A.

Much is known about A-hypergeometric D-modules. They are holonomic for all parameters [GKZ89, Ado94], and they are regular holonomic exactly when I_A is $\mathbb{Z}$-graded in the usual sense [Hot91, SW08]. In this case, (Gamma-)series pexpansions for the solutions of $H_A(\beta)$ centered at the origin and convergent in certain domains can be explicitly computed [GKZ89, SST00]. The generic (minimal) holonomic rank is known to be $\operatorname{vol}(A)$, the normalized volume of the convex hull of the columns of A and the origin [GKZ89, Ado94], and holonomic rank is independent of the parameter β if and only if the semigroup ring $\mathbb{C}[\mathbb{N}A]$ is Cohen-Macaulay [GKZ89, Ado94, MMW05]. We will extend all of these results, suitably modified, to the general setting of binomial D-modules. The important caveat is that a general binomial D-module can exhibit behavior that is forbidden to GKZ systems.

Example 1.7. Let $A = \left[\begin{smallmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{smallmatrix}\right]$ and let B be given by its transpose $B^t = \left[\begin{smallmatrix} 1 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{smallmatrix}\right]$, so that

$$H(B, \beta) = \langle \partial_1\partial_3 - \partial_2, \partial_1\partial_4 - \partial_2 \rangle + \langle x_1\partial_1 - x_2\partial_2 - \beta_1, x_2\partial_2 + x_3\partial_3 + x_4\partial_4 - \beta_2 \rangle.$$

If $\beta_1 = 0$, then any (local holomorphic) bivariate function $f(x_3, x_4)$ annihilated by the operator $x_3\partial_3 + x_4\partial_4 - \beta_2$ is a solution of $H(B, \beta)$. The space of such functions has uncountable dimension, as it contains all monomials $x_3^{w_3}x_4^{w_4}$ with $w_3, w_4 \in \mathbb{C}$ and $w_3 + w_4 = \beta_2$.

We also wish to understand whether series solutions of a Horn system (or binomial D-module) are *fully supported*, which means that there is an unbounded strictly convex rational polyhedron of dimension m (the maximum possible) whose lattice points correspond to monomials with nonzero coefficients. Generally speaking, Horn systems in dimension $m \geq 2$ tend to have many series solutions without full support.

Example 1.8. Given any $\beta \in \mathbb{C}^2$ and $A = \left[\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{smallmatrix}\right]$, $B^t = \left[\begin{smallmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{smallmatrix}\right]$, the Puiseux monomial $x_1^{\beta_1 - \beta_2/3} x_4^{\beta_2/3}$ is a solution of $H(B, \beta)$. A key feature of this example is that the solutions without full support persist for arbitrary choices of the parameter vector β. This phenomenon was originally noticed in [Erd50].

The purpose of this article is to answer the following completely and precisely.

Questions 1.9. *Fix B as in Convention 1.2 and consider the Horn systems determined by B.*

1. *For which parameters does the space of local holomorphic solutions around a nonsingular point have finite dimension as a complex vector space?*
2. *What is a combinatorial formula for the minimum such dimension, over all possible choices of parameters?*
3. *Which parameters are generic, in the sense that the minimum dimension is attained?*
4. *How do (the supports of) series solutions centered at the origin look, combinatorially?*

These questions make sense for classical Horn systems and binomial Horn systems, and also for binomial D-modules. In this language, we can also answer the following.

Questions 1.9 (continued)**.** *Consider the binomial D-modules $H_A(I, \beta)$ for varying $\beta \in \mathbb{C}^d$.*

5. *When is $D/H_A(I, \beta)$ a holonomic D-module?*
6. *When is $D/H_A(I, \beta)$ a* regular *holonomic D-module?*

The phenomena underlying all of the answers to Questions 1.9 are controlled by the primary decomposition of the binomial ideal I. We recall some terminology from [ES96] and [DMM08], our main references on binomial primary decomposition. A binomial prime ideal $I_{\rho,J}$ in $\mathbb{C}[\partial_1, \ldots, \partial_n]$ is determined by a subset $J \subseteq \{1, \ldots, n\}$ and a character $\rho : L \to \mathbb{C}^*$ for some saturated sublattice $L \subseteq \mathbb{Z}^J$. We say that the sublattice $L \subseteq \mathbb{Z}^J$ is *associated* to B when $I_{\rho,J}$ is an associated prime of the ideal $I(B)$; the multiplicity $\mu(L, J)$ of L in $I(B)$ is $|L/(\mathbb{Z}B \cap \mathbb{Z}^J)|$ times the multiplicity of $I_{\rho,J}$ in $I(B)$. The factor $|L/(\mathbb{Z}B \cap \mathbb{Z}^J)|$ counts the number of partial characters $\rho : L \to \mathbb{C}^*$ for which $I_{\rho,J}$ is associated to $I(B)$.

Example 1.10. [Example 1.8, continued] The binomial Horn system is

$$H(B, \beta) = I(B) + \langle x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4 - \beta_1, x_2\partial_2 + 2x_3\partial_3 + 3x_4\partial_4 - \beta_2 \rangle \subseteq D_4.$$

The primary decomposition of the lattice basis ideal $I(B)$ in $\mathbb{C}[\partial_1, \partial_2, \partial_3, \partial_4]$ is

$$I(B) = \langle \partial_1\partial_3 - \partial_2^2, \partial_2\partial_4 - \partial_3^2 \rangle = \langle \partial_1\partial_3 - \partial_2^2, \partial_2\partial_4 - \partial_3^2, \partial_1\partial_4 - \partial_2\partial_3 \rangle \cap \langle \partial_2, \partial_3 \rangle.$$

The first of these components is the toric ideal $I_A = I_{\rho,J}$ of the twisted cubic curve, where $\rho : \ker(A) = \mathbb{Z}B \to \mathbb{C}^*$ is the trivial character and $J = \{1, 2, 3, 4\}$. The ideal $\langle \partial_2, \partial_3 \rangle$ is the binomial prime ideal $I_{\rho,J}$ for the (automatically) trivial character $\rho : \mathbf{0} \to \mathbb{C}^*$ and the subset $J = \{1, 4\}$. Both of these ideals have multiplicity 1 in $I(B)$, which is a radical ideal. This explains the associated lattices and multiplicities in Example 1.15 below.

Definition 1.11. An associated saturated sublattice $L \subseteq \mathbb{Z}^J \cap \ker(A)$ is called *toral* if $L = \mathbb{Z}^J \cap \ker(A)$; otherwise, $L \subsetneq \mathbb{Z}^J \cap \ker(A)$ is called *Andean*.

Example 1.12. [Example 1.7 continued] With A and B as in Example 1.7, there are two associated lattices, one with $J = \{1,2,3,4\}$, the other with $J = \{3,4\}$. The first one is toral, while the second is Andean.

Andean sublattices, when they exist, provoke failure of holonomicity in a basic way, by producing infinite dimensional solution spaces, as in Example 1.7. This behavior is controlled by the *Andean arrangement*, which we illustrate below.

Example 1.13. [Example 1.12 continued] The Andean arrangement in this case is $\mathbb{C}\left[\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}\right] = \left\{ \left[\begin{smallmatrix} 0 \\ \beta_2 \end{smallmatrix}\right] : \beta_2 \in \mathbb{C} \right\}$. As we have already checked, the Horn system in Example 1.7 is not holonomic for this set of parameters. It will follow from our results that these are the only parameters where holonomicity fails.

The answers to Questions 1.9 are as follows.

Answers 1.14.

1. *(Theorem 6.3) The dimension is finite exactly for $-\beta$ not in the Andean arrangement.*
2. *(Theorem 6.10) The generic (minimum) rank is $\sum \mu(L,J)\cdot\mathrm{vol}(A_J)$, the sum being over all toral associated sublattices with $\mathbb{C}A_J = \mathbb{C}^d$, where $\mathrm{vol}(A_J)$ is the volume of the convex hull of A_J and the origin, normalized so a lattice simplex in $\mathbb{Z}A_J$ has volume* 1.
3. *(Definition 6.9 and Theorem 6.10) The minimum rank is attained precisely when $-\beta$ lies outside of an affine subspace arrangement determined by certain local cohomology modules, with the same flavor as (and containing) the Andean arrangement.*
4. *(Theorem 6.10, Theorem 7.14, and Corollary 7.25) When the Horn system is regular holonomic and β is general, there are $\mu(L,J)\cdot\mathrm{vol}(A_J)$ linearly independent Puiseux series solutions supported on (translates of) L-bounded components, with coefficients determined by hypergeometric recursions. Only $g\cdot\mathrm{vol}(A)$ many Gamma-series solutions have full support, where $g = |\ker(A)/\mathbb{Z}B|$ is the index of $\mathbb{Z}B$ in its saturation.*
5. *(Theorem 6.3)Holonomicity is equivalent to the finite dimension in Answer 1.14.1.*
6. *(Theorem 6.3)Holonomicity is equivalent to regular holonomicity when I is standard $\mathbb{Z}$-graded—i.e., the row-span of A contains the vector $[1\cdots 1]$. Conversely, if there exists a parameter β for which $D/H_A(I,\beta)$ is regular holonomic, then I is $\mathbb{Z}$-graded.*

In Answer 1.14.4, the solutions for toral sublattices $L = \ker(A)\cap\mathbb{Z}^J$ in which J is a proper subset of $\{1,\dots,n\}$ give rise to solutions that are are supported on sets of dimension $\mathrm{rank}(L) = |J| - d < n - d = m$. Answer 1.14.6 is, given the other results in this paper, an (easy) consequence of the (hard) holonomic regularity results of Hotta [Hot91] and Schulze–Walther [SW08]. Finally, let us note again that most of the theorems quoted in Answers 1.14 are stated and proved in the context of arbitrary binomial D-modules, not just Horn systems. This generality is forced upon us, even if the interest is solely in Horn systems, because the binomial ideals that arise in the course of primary decomposition of lattice basis ideals are more or less arbitrary.

We concentrate on the special case of Horn systems in Section 7. The systematic study of binomial Horn systems was started in [DMS05] under the hypothesis that m (the number of columns of B) is equal to 2. See also [Sad02].

We illustrate Answers 1.14 in our running example.

Example 1.15. [Examples 1.8 and 1.10, continued] There are two associated sublattices $L \subseteq \mathbb{Z}^J$ here, both toral, and both satisfying $\mathbb{C}A_J = \mathbb{C}^2$: the sublattice $\ker(A) \subseteq \mathbb{Z}^4$, where $J = \{1,2,3,4\}$, and the sublattice $\mathbf{0} \subseteq \mathbb{Z}^J$ for $J = \{1,4\}$. Both of the multiplicities $\mu\big(\ker(A), \{1,2,3,4\}\big)$ and $\mu\big(\mathbf{0}, \{1,4\}\big)$ equal 1, while $\mathrm{vol}(A) = 3$ and $\mathrm{vol}(A_{\{1,4\}}) = 1$, the latter because $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]$ form a basis for the lattice they generate. Hence, for generic parameters, there are four solutions in total, three of them with full support and one—namely the Puiseux monomial in Example 1.8—with support of dimension zero. The solution space of this system is always four-dimensional, but if the parameters are not generic, the solutions do not arise directly from the associated lattices; see Example 7.23 for more details. The intersection of the two irreducible varieties in the zero set of $I(B)$ in $\mathbb{C}^4$ is the zero set of

$$\langle \partial_1\partial_3 - \partial_2^2, \partial_2\partial_4 - \partial_3^2, \partial_1\partial_4 - \partial_2\partial_3 \rangle + \langle \partial_2, \partial_3 \rangle = \langle \partial_1\partial_4, \partial_2, \partial_3 \rangle.$$

In this case, the primary arrangement from Theorem 6.8 is $\mathbb{C}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \cup \mathbb{C}\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]$. When β lies off these two lines, Theorem 6.8 yields an isomorphism of D_4-modules:

$$\frac{D_4}{H(B,\beta)} \cong \frac{D_4}{\langle \partial_1\partial_3 - \partial_2^2, \partial_2\partial_4 - \partial_3^2, \partial_1\partial_4 - \partial_2\partial_3 \rangle + \langle E - \beta \rangle} \oplus \frac{D_4}{\langle \partial_2, \partial_3 \rangle + \langle E - \beta \rangle}.$$

The summands on the right-hand side are GKZ hypergeometric systems (up to extraneous vanishing variables in the $\langle \partial_2, \partial_3 \rangle$ case) with holonomic ranks 3 and 1, respectively.

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2. THE EULER-KOSZUL FUNCTOR

The Euler operators in Definition 1.6 can be used to build a Koszul-like complex whose zeroth homology is a binomial D-module. In its most basic form, this construction is due to Gelfand, Kapranov, and Zelevinsky [GKZ89], and was developed by Adolphson [Ado94, Ado99] and Okuyama [Oku06]. A functorial generalization was introduced in [MMW05], where it was proved to be homology-isomorphic to an ordinary Koszul complex detecting holonomic rank changes for varying parameters β. We review the definitions and connection to the quasidegrees from [MMW05, Sections 4 and 5].

Given a matrix A with columns $a_1, \ldots, a_n$ as in Convention 1.3, recall that the polynomial ring $\mathbb{C}[\partial]$ and the Weyl algebra $D = D_n$ are A-graded by $\deg(\partial_j) = -a_j$ and $\deg(x_j) = a_j$. Under this A-grading, operators $E_1, \ldots, E_d$, and in fact all of the products $x_j\partial_j \in D$, are homogeneous of degree 0.

Given an A-graded left D-module W, if $z \in W_\alpha$ is homogeneous of degree α then set $\deg_i(z) = \alpha_i$. The map $E_i - \beta_i : W \to W$ that sends each homogeneous element $z \in W$ to

$$(E_i - \beta_i) \circ z = (E_i - \beta_i - \deg_i(z))z, \tag{2.1}$$

and is extended $\mathbb{C}$-linearly to all of W, determines a D-linear endomorphism of W.

Definition 2.1. Fix $\beta \in \mathbb{C}^d$ and an A-graded $\mathbb{C}[\partial]$-module V. The *Euler-Koszul complex* $\mathcal{K}_\bullet(E - \beta; V)$ is the Koszul complex of left D-modules defined by the sequence $E - \beta$ of commuting endomorphisms on the left D-module $D \otimes_{\mathbb{C}[\partial]} V$ concentrated in homological degrees d to 0. The i^{th} *Euler-Koszul homology* of V is $\mathcal{H}_i(E - \beta; V) = H_i(\mathcal{K}_\bullet(E - \beta; V))$.

Example 2.2. Fix A and B as in Convention 1.4.

1. The *binomial Horn D-module* with parameter β is $\mathcal{H}_0(E - \beta; \mathbb{C}[\partial]/I(B))$.
2. The *A-hypergeometric D-module* with parameter β is $\mathcal{H}_0(E - \beta; \mathbb{C}[\partial]/I_A)$; see (1.2).
3. If $I \subseteq \mathbb{C}[\partial]$ is any A-graded binomial ideal, then $\mathcal{H}_0(E - \beta; \mathbb{C}[\partial]/I) = D/H_A(I, \beta)$.

Euler-Koszul homology behaves predictably with regard to A-graded translation.

Lemma 2.3. *Let V be an A-graded $\mathbb{C}[\partial]$-module and $\alpha \in \mathbb{Z}^d = \mathbb{Z}A$. If $V(\alpha)$ is the A-graded module with $V(\alpha)_{\alpha'} = V_{\alpha+\alpha'}$, then $\mathcal{H}_0(E - \beta; V(\alpha)) \cong \mathcal{H}_0(E - \beta + \alpha; V)(\alpha)$.* □

We shall see that Euler-Koszul homology has the useful property of detecting "where" a module is nonzero, the nonzeroness being measured in the following sense.

Definition 2.4. Let V be an A-graded $\mathbb{C}[\partial]$-module. The set of *true degrees* of V is

$$\operatorname{tdeg}(V) = \{\beta \in \mathbb{Z}^d : V_\beta \neq 0\}.$$

The set $\operatorname{qdeg}(V)$ of *quasidegrees* of V is the Zariski closure in $\mathbb{C}^d$ of its true degrees $\operatorname{tdeg}(V)$.

Because of the next lemma, we shall often refer to quasidegree sets as *arrangements*.

Lemma 2.5. *Let R be a noetherian A-graded ring that is finitely generated over its degree 0 piece. The quasidegree set of any finitely generated graded R-module is a finite union of affine subspaces of $\mathbb{C}^d$, each spanned by the degrees of some subset of the generators of R.*

Proof. Every A-graded module has an A-graded associated prime, and therefore a submodule isomorphic to an A-graded translate of a quotient by an A-graded prime. Now use Noetherian induction to conclude that every such module has a filtration whose successive quotients are A-translates of quotients of R modulo prime ideals. But being an integral domain, the true degree set of a quotient $R/\mathfrak{p}$ by a prime ideal $\mathfrak{p}$ is the affine semigroup generated by the degrees of the generators of R that remain nonzero in $R/\mathfrak{p}$. □

Example 2.6. Let $I = \langle bd - de, bc - ce, ab - ae, c^3 - ad^2, a^2d^2 - de^3, a^2cd - ce^3, a^3d - ae^3\rangle$ be a binomial ideal in $\mathbb{C}[\partial]$, where we write $\partial = (\partial_1, \partial_2, \partial_3, \partial_4, \partial_5) = (a, b, c, d, e)$, and let

$$A = \left[\begin{smallmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 1 \end{smallmatrix}\right] \qquad \text{and} \qquad B^t = \begin{bmatrix} -2 & 3 & 0 & -1 & 0 \\ -1 & 0 & 3 & -2 & 0 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix}.$$

One easily verifies that the binomial ideal I is graded by $\mathbb{Z}A = \mathbb{Z}^2$. If ω is a primitive cube root of unity ($\omega^3 = 1$), then I, which is a radical ideal, has the prime decomposition

$$I = \langle a,c,d\rangle \cap \langle bc-ad, b^2-ac, c^2-bd, b-e\rangle \cap \langle \omega bc - ad, b^2-\omega ac, \omega^2 c^2 - bd, b-e\rangle \\ \cap \langle \omega^2 bc - ad, b^2 - \omega^2 ac, \omega c^2 - bd, b-e\rangle.$$

If $V = \mathbb{C}[a,b,c,d,e]/\langle a,c,d\rangle$, then $\mathrm{qdeg}(V)$ is the diagonal line in $\mathbb{C}^2$. In contrast, the quotient by each one of the other three prime ideals there has quasidegree set equal to all of $\mathbb{C}^2$. It follows that $\mathrm{qdeg}(\mathbb{C}[\partial]/I) = \mathbb{C}^2$.

Let $\mathfrak{m}$ be the maximal ideal $\langle \partial_1, \ldots \partial_n\rangle$ of $\mathbb{C}[\partial]$. Since A is pointed with no nonzero columns, $\mathfrak{m}$ is the unique maximal A-graded ideal. Given an A-graded $\mathbb{C}[\partial]$-module V, its *local cohomology modules*

$$H^i_{\mathfrak{m}}(V) = \varinjlim_t \mathrm{Ext}^i_{\mathbb{C}[\partial]}(\mathbb{C}[\partial]/\mathfrak{m}^t, V)$$

supported at $\mathfrak{m}$ are A-graded; see [MS05, Chapter 13]. Even when V is finitely generated, its local cohomology modules $H^i_{\mathfrak{m}}(V)$ need not be; but their Matlis duals are, so their quasidegree sets are still arrangements.

Lemma 2.7. *If V is a finitely generated A-graded $\mathbb{C}[\partial]$-module, then the quasidegree set $\mathrm{qdeg}(H^i_{\mathfrak{m}}(V))$ of the i^{th} local cohomology module of V is a union of finitely many integer translates of the complex subspaces $\mathbb{C}A_J \subseteq \mathbb{C}^n$ spanned by $\{a_j : j \in J\}$ for various J.*

Proof. Let $\varepsilon_A = -\sum_{j=1}^n a_j$. In the graded version [Mil02b, Theorem 6.3] of the Greenlees-May theorem [GM92], setting $\mathcal{E}$ equal to the injective hull of the residue field $\mathbb{C}[\partial]/\mathfrak{m}$ yields the natural A-graded local duality vector space isomorphism

$$\mathrm{Ext}^{n-i}_{\mathbb{C}[\partial]}(V, \mathbb{C}[\partial])_\alpha \cong \mathrm{Hom}_{\mathbb{C}}(H^i_{\mathfrak{m}}(V)_{-\alpha+\varepsilon_A}, \mathbb{C}).$$

(Use the case $\mathcal{G} = \mathbb{C}[\partial]$ to deduce that the right-hand side of [Mil02b, Theorem 6.3] is the derived Hom into the canonical module $\omega_{\mathbb{C}[\partial]}$, which is isomorphic as a graded module to the principal ideal $\langle \partial_1 \cdots \partial_n\rangle$; see also [BH93, Section 3.5] for $\mathbb{Z}$-graded local duality.) Hence $\varepsilon_A + \mathrm{qdeg}(H^i_{\mathfrak{m}}(V)) = -\mathrm{qdeg}(\mathrm{Ext}^{n-i}_{\mathbb{C}[\partial]}(V, \mathbb{C}[\partial]))$ is the negative of the quasidegree set of a finitely generated module. The result is now a consequence of Lemma 2.5. □

3. Binomial primary decomposition

In this section we review some prerequisites on primary decomposition of binomial ideals from [ES96] and [DMM08], including interactions with A-gradings. For the applications to Horn D-modules in Section 7, we pay special attention to lattice basis ideals. For the duration of this section we work over a polynomial ring $\mathbb{C}[\partial]$ in commuting variables $\partial = \partial_1, \ldots, \partial_n$.

If $L \subseteq \mathbb{Z}^n$ is a sublattice, then the *lattice ideal* of L is $I_L = \langle \partial^{u_+} - \partial^{u_-} : u = u_+ - u_- \in L\rangle$. Here and henceforth, u_+ has i^{th} coordinate u_i if $u_i \geq 0$ and 0 otherwise. The vector $u_- \in \mathbb{N}^q$ is defined by $u_+ - u_- = w$, or equivalently, $u_- = (-u)_+$. More general than I_L are the ideals

$$I_\rho = \langle \partial^{u_+} - \rho(u)\partial^{u_-} : u = u_+ - u_- \in L\rangle$$

for any *partial character* $\rho : L \to \mathbb{C}^*$ of $\mathbb{Z}^n$, which includes the data of both its domain lattice $L \subseteq \mathbb{Z}^n$ and the map to $\mathbb{C}^*$. (The ideal I_ρ is called $I_+(\rho)$ in [ES96].) The ideal I_ρ is prime if and only if L is a *saturated* sublattice of $\mathbb{Z}^n$, meaning that L equals its *saturation*

$$\mathrm{sat}(L) = (\mathbb{Q}L) \cap \mathbb{Z}^n,$$

where $\mathbb{Q}L = \mathbb{Q} \otimes_{\mathbb{Z}} L$ is the rational vector space spanned by L in $\mathbb{Q}^n$. In fact [ES96, Corollary 2.6], every binomial prime ideal in $\mathbb{C}[\partial]$ has the form

$$I_{\rho,J} = I_\rho + \langle \partial_j : j \notin J \rangle$$

for some saturated partial character ρ (i.e., whose domain is a saturated sublattice) and subset $J \subseteq \{1, \ldots, n\}$ such that the binomial generators of I_ρ only involve variables ∂_j for $j \in J$ (some of which might actually be absent from the generators of I_ρ).

Example 3.1. The intersectand $\langle a, c, d \rangle$ in Example 2.6 equals the prime ideal $I_{\rho,J}$ for $J = \{2, 5\}$ and $L = \{0\} \subseteq \mathbb{Z}^J$. The remaining three intersectands are the prime ideals $I_{\rho,J}$ for the three characters ρ that are defined on $\ker(A)$ but trivial on its index 3 sublattice $\mathbb{Z}B$ spanned by the columns of B, where $J = \{1, 2, 3, 4, 5\}$.

Theorem 3.2 ([DMM08, Theorem 3.2])**.** *Fix a binomial ideal I. Write ∂_J for the monomial $\prod_{j \in J} \partial_j$. Each associated prime $I_{\rho,J}$ has an explicitly defined monomial ideal $U_{\rho,J}$ such that*

$$I = \bigcap_{I_{\rho,J} \in \mathrm{Ass}(I)} \mathcal{C}_{\rho,J} \quad \textit{for} \quad \mathcal{C}_{\rho,J} = \big((I + I_\rho) : \partial_J^\infty\big) + U_{\rho,J}$$

is a decomposition of I as an intersection of primary binomial ideals.

It is not important for our present purposes precisely what $U_{\rho,J}$ is in general; all we need are various consequences, especially for the structure of the quotients $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$, derived in [DMM08] from the explicit description. The flavor is captured in the following example and in Example 3.7, where the precise answer for certain minimal primes is quite clean.

Example 3.3. Fix matrices A and B as in Convention 1.4. This identifies $\mathbb{Z}^d$ with the quotient of $\mathbb{Z}^n/\mathbb{Z}B$ modulo its torsion subgroup. The *lattice basis ideal* corresponding to the lattice $\mathbb{Z}B = \{Bz : z \in \mathbb{Z}^m\}$ is defined by

$$I(B) = \langle \partial^{u_+} - \partial^{u_-} : u = u_+ - u_- \text{ is a column of } B \rangle \subseteq \mathbb{C}[\partial_1, \ldots, \partial_n].$$

Each of the minimal primes of $I(B)$ arises, after row and column permutations, from a block decomposition of B of the form

$$\left[\begin{array}{c|c} N & B_J \\ \hline M & 0 \end{array}\right], \tag{3.1}$$

where M is a mixed submatrix of B of size $q \times p$ for some $0 \le q \le p \le m$ [HS00]. (Matrices with $q = 0$ rows are automatically mixed; matrices with $q = 1$ row are never mixed.) We note that not all such decompositions correspond to minimal primes: the matrix M has to satisfy another condition which Hoşten and Shapiro call irreducibility [HS00, Definition 2.2 and Theorem 2.5]. If $I(B)$ is a complete intersection, then only square matrices M will appear in the block decompositions (3.1), by a result of Fischer and Shapiro [FS96].

For each partial character $\rho : \mathrm{sat}(\mathbb{Z}B_J) \to \mathbb{C}^*$ extending the trivial character on $\mathbb{Z}B_J$, the ideal $I_{\rho,J}$ is associated to $I(B)$, where $J = J(M) = \{1, \ldots, n\} \smallsetminus \mathrm{rows}(M)$ indexes the $n - q$ rows not in M. Henceforth, we denote $\bar{J} = \{1, \ldots, n\} \smallsetminus J$, which in this case means that $\bar{J} = \mathrm{rows}(M)$. We reiterate that the symbol ρ here includes the specification of the sublattice $\mathrm{sat}(\mathbb{Z}B_J) \subseteq \mathbb{Z}^n$. The corresponding primary component $\mathcal{C}_{\rho,J}$ of $I(B)$ is simply I_ρ if $q = 0$, but will in general be non-radical when $q \geq 2$ (recall that $q = 1$ is impossible).

Since A-gradings are central to our theory, we collect some relevant results from [DMM08]. Recall Convention 1.3. Henceforth, A_J denotes the submatrix of A whose columns are indexed by J. We write $\mathbb{Z}A_J \subseteq \mathbb{Z}^d = \mathbb{Z}A$ for the group generated by these columns.

Lemma 3.4. *Fix a partial character $\rho : L \to \mathbb{C}^*$ for a saturated sublattice $L \subseteq \mathbb{Z}^J \subseteq \mathbb{Z}^n$. Let $\mathcal{C}_{\rho,J}$ be an A-graded binomial $I_{\rho,J}$-primary ideal. Then $L \subseteq \mathbb{Z}^J \cap \ker_{\mathbb{Z}}(A) = \ker_{\mathbb{Z}}(A_J)$, the Krull dimension satisfies* $\dim(\mathbb{C}[\partial]/I_{\rho,J}) \geq \mathrm{rank}(A_J)$, *and the following are equivalent.*

- *The* Hilbert function $\mathbb{Z}A \to \mathbb{N}$ *defined by* $\alpha \mapsto \dim_{\mathbb{C}}(\mathbb{C}[\partial]/\mathcal{C}_{\rho,J})_\alpha$ *is bounded above.*
- *The homomorphism $\mathbb{Z}^J/L \twoheadrightarrow \mathbb{Z}A_J \subseteq \mathbb{Z}^d$ is injective.*
- $L = \ker_{\mathbb{Z}}(A_J)$.
- $\dim(\mathbb{C}[\partial]/I_{\rho,J}) = \mathrm{rank}(A_J)$.

When these conditions are satisfied, the module $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ and the lattice L are called toral*, the ideal $I_{\rho,J}$ is called a* toral prime*, and $\mathcal{C}_{\rho,J}$ is called a* toral (primary) component*. When these conditions are not satisfied, substitute* Andean *(see Remark 5.3) for "toral" above.*

Proof. These conditions are the ones appearing, respectively, in [DMM08, Definition 4.3, Proposition 4.7, Corollary 4.8, and Lemma 4.9]. □

Example 3.5. In Example 3.1, the homomorphism $A_{\{2,5\}} : \mathbb{Z}^{\{2,5\}} \to \mathbb{Z}^2$ is not injective since it maps both basis vectors to $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$; thus the prime ideal $\langle a, c, d\rangle$ is an Andean component of I. In contrast, the remaining associated prime ideals are all toral by Lemma 3.4, with $A_J = A$.

The final example in this section demonstrates, at long last, just how concrete binomial primary decomposition can be when expressed in combinatorial terms. It will be applied directly in Section 7 to construct solutions to binomial Horn systems. Example 3.7 was, for us, the motivation and starting point for all of the other results in this article and in [DMM08].

Definition 3.6. Any integer matrix M with q rows defines an undirected graph $\Gamma(M)$ having vertex set $\mathbb{N}^q$ and an edge from u to v if $u - v$ or $v - u$ is a column of M. An *M-subgraph* of $\mathbb{N}^q$ is a connected component of $\Gamma(M)$. An M-subgraph is *bounded* if it has finitely many vertices, and *unbounded* otherwise. (See Example 7.8 for an explicit computation in $\mathbb{N}^3$.)

Example 3.7. Resume the notation of Example 3.3. If $I_{\rho,J}$ is a toral minimal prime of $I(B)$ given by a matrix decomposition as in (3.1), then

$$\mathcal{C}_{\rho,J} = I(B) + I_\rho + U_M,$$

where $U_M \subseteq \mathbb{C}[\partial_j : j \in \bar{J}]$ is the ideal $\mathbb{C}$-linearly spanned by all monomials with exponent vectors in the union of the unbounded M-subgraphs of $\mathbb{N}^{\bar{J}}$; this is [DMM08, Corollary 4.14], which also says that every monomial in $\mathcal{C}_{\rho,J}$ already lies in U_M.

4. Toral modules

Much of this article concerns widely diverging D-module theoretic behavior lifted from the toral vs. Andean dichotomy in the primary components of graded binomial ideals. The functor translating to D-modules is Euler-Koszul homology, which was originally conceived of for *toric* modules [MMW05, Definition 4.5]. Here, we shall show that all of the main results in [MMW05] hold, with essentially the same proofs, for the more general class of *toral* modules in Definition 4.1. The key starting point is the filtration characterization in Proposition 4.2. Our main results for toral modules are Theorems 4.5, 4.6, 4.8, and 4.9.

Definition 4.1. An A-graded $\mathbb{C}[\partial]$-module V is *natively toral* if there is a binomial prime ideal $I_{\rho,J}$ and a degree $\alpha \in \mathbb{Z}^d$ such that $V(\alpha) \cong \mathbb{C}[\partial]/I_{\rho,J}$ is a toral quotient (Lemma 3.4). The module V is *toral* if it is finitely generated and its A-graded Hilbert function is bounded.

Proposition 4.2. *An A-graded $\mathbb{C}[\partial]$-module V is toral if and only if it has a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = V$ whose successive quotients V_k/V_{k-1} are all natively toral.*

Proof. The proof proceeds by Noetherian induction to reduce to the prime case, and then by showing that every toral prime is binomial. The argument is the same as for [DMM08, Proposition 4.7], but with general modules V in place of primary quotients $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$. □

The argument in the proof of Proposition 4.2 actually shows more.

Lemma 4.3. *If $W \subseteq V$ are A-graded modules with V toral, then W and V/W are toral.*

Proof. Intersecting any toral filtration of V with W yields a filtration of W whose successive quotients are toral because they are A-graded modules over natively toral quotients $\mathbb{C}[\partial]/I_{\rho,J}$. Hence W is toral. The same argument works for the image filtration in V/W. □

We begin recounting the results of [MMW05] with an elementary observation about how Euler-Koszul homology works for modules killed by some of the variables; the proof is the same as [MMW05, Lemma 4.8]. For notation, let E_i^J be the operator obtained from E_i by setting the terms $x_j\partial_j$ to zero for $j \notin J$. This operator can be thought of as lying in the Weyl algebra D_J in the variables x_j and ∂_j for $j \in J$. Denote by $x_{\bar{J}}$ the x-variables for $j \notin J$.

Lemma 4.4. *If the variables ∂_j for $j \notin J$ annihilate an A-graded $\mathbb{C}[\partial]$-module V, then $D \otimes_{\mathbb{C}[\partial]} V \cong \mathbb{C}[x_{\bar{J}}] \otimes_{\mathbb{C}} (D_J \otimes_{\mathbb{C}[\partial_J]} V)$ as $D = D_{\bar{J}} \otimes_{\mathbb{C}} D_J$-modules. Acting by E_i on $D \otimes_{\mathbb{C}[\partial]} V$ as in (2.1) is the same as acting by E_i^J on the right-hand factor of $\mathbb{C}[x_{\bar{J}}] \otimes_{\mathbb{C}} (D_J \otimes_{\mathbb{C}[\partial_J]} V)$.* □

Many of the following results are stated in the context of *holonomic* D-modules, which by definition are the finitely generated left D-modules W with $\mathrm{Ext}_D^j(W, D) = 0$ for $j \neq n$. When W is holonomic, the vector space $\mathbb{C}(x) \otimes_{\mathbb{C}[x]} W$ over the field $\mathbb{C}(x)$ of rational functions in $x_1, \ldots, x_n$ has finite dimension equal to the *holonomic rank* $\mathrm{rank}(W)$ by a celebrated theorem of Kashiwara; see [SST00, Theorem 1.4.19 and Corollary 1.4.14].

We shall also be interested in whether our D-modules are *regular holonomic* (see [Bjö79] for a definition). For an A-hypergeometric D-module (Example 2.2), regular holonomicity is known [Hot91] to occur when A is *homogeneous*, meaning that there is a row vector $\psi \in \mathbb{Q}^d$

such that ψA equals the row vector $[1, \dots, 1]$. In this case, the $\mathbb{Z}A = \mathbb{Z}^d$-grading on $\mathbb{C}[\partial]$ coarsens naturally to the *standard $\mathbb{Z}$-grading*, in which $\deg(\partial_j) = 1 \in \mathbb{Z}$ for all j.

Theorem 4.5. *If V is a toral $\mathbb{C}[\partial]$-module and $\beta \in \mathbb{C}^d$, then the Euler-Koszul homology $\mathcal{H}_i(E - \beta; V)$ is holonomic for all i. Moreover, the following are equivalent.*

1. *$\mathcal{H}_0(E - \beta; V)$ has holonomic rank 0.*
2. *$\mathcal{H}_0(E - \beta; V) = 0$.*
3. *$\mathcal{H}_i(E - \beta; V) = 0$ for all $i \geq 0$.*
4. *$-\beta \notin \operatorname{qdeg}(V)$.*

If, in addition, the matrix A is homogeneous, then $\mathcal{H}_i(E-\beta; V)$ is regular holonomic for all i.

Proof. This is the toral generalization of [MMW05, Proposition 5.1] and [MMW05, Proposition 5.3]. To see that it holds, start with [MMW05, Notation 4.4]: instead of only allowing submatrices of A corresponding to faces of the semigroup $\mathbb{N}A$, we allow submatrices A_J with arbitrary column sets $J \subseteq \{1, \dots, n\}$. Then, in [MMW05, Definition 4.5], replace "toric" with "toral" and change S_{F_k} to $\mathbb{C}[\partial]/I_{\rho,J}$; that this defines toral modules is by Lemma 3.4.

The key is [MMW05, Lemma 4.9]. In the proof there, first replace $\mathcal{M}_\beta^F = D/H_A(I_A^F, \beta)$ by $D/H_A(I_{\rho,J}, \beta)$. Then observe that rescaling the variables via ρ induces an A-graded automorphism of D commuting with the construction of Euler-Koszul complexes (because $x_j\partial_j$ is invariant under the automorphism). Hence the theorem for natively toral modules need only be proved in the special case $\rho =$ identity. This allows us to use $I_{A_J} + \langle \partial_j : j \notin J \rangle$ instead of $I_{\rho,J}$. The rest of the proof of [MMW05, Lemma 4.9] goes through unchanged, and when A is homogeneous, provides regular holonomicity as a consequence of the analogous result for GKZ systems from [Hot91, SW08].

Now extend the proof of [MMW05, Proposition 5.1] to the toral setting. For the first paragraph of that proof, replace "toric" with "toral" and replace S_F by $\mathbb{C}[\partial]/I_{\rho,J}$. For the later paragraphs of the proof, begin by working with the module M there being native toral. This allows us to replace I_A, when it arises as an annihilator toward the end, with $I_{\rho,J}$, thereby proving the native toral case. For the arbitrary toral case, simply note that for any exact sequence $0 \to V' \to V \to V'' \to 0$ in which V' and V'' both have (regular) holonomic Euler-Koszul homology, each Euler-Koszul homology module of V is placed between two (regular) holonomic modules, and is hence (regular) holonomic.

Finally, to generalize [MMW05, Proposition 5.3], replace "toric" with "toral" in the statement and proof. Then, in the proof, replace I_A^F by $I_{\rho,J}$ and S_F by $\mathbb{C}[\partial]/I_{\rho,J}$. □

Next we record the toral generalization of [MMW05, Theorem 6.6].

Theorem 4.6. *The Euler-Koszul homology $\mathcal{H}_j(E - \beta; V)$ of a toral module V is nonzero for some $j \geq 1$ if and only if $-\beta \in \operatorname{qdeg}(H^i_{\mathfrak{m}}(V))$ for some $i < d$. More precisely, if k equals the smallest homological degree i for which $-\beta \in \operatorname{qdeg}(H^i_{\mathfrak{m}}(V))$, then $\mathcal{H}_{d-k}(E - \beta; V)$ is holonomic of nonzero rank while $\mathcal{H}_j(E - \beta; V) = 0$ for $j > d - k$.*

Proof. Begin by noting that $\operatorname{Ext}^i_{\mathbb{C}[\partial]}(V, \mathbb{C}[\partial])$ is toral whenever V is toral. This is the toral generalization of [MMW05, Lemma 6.1]; the same proof works, mutatis mutandis, replacing

S_A in [MMW05] by $\mathbb{C}[\partial]/I_{\rho,J}$ here. Now extend [MMW05, Theorem 6.3] to the toral case: the only property of toric modules used in its proof is the holonomicity of Euler-Koszul homology, which we have shown is true for toral modules in Theorem 4.5. Finally, to torally extend the toric [MMW05, Theorem 6.6], start with the first sentence of the proof, which for toral modules is Lemma 4.7, below. After that, the proof goes through verbatim, given that we have shown the results it cites for toric modules to be true for toral modules. □

Lemma 4.7. *If V is toral, then its Krull dimension satisfies* $\dim(V) = \dim(\mathrm{qdeg}(V)) \leq d$.

Proof. For natively toral modules this follows from Lemma 3.4. For arbitrary toral modules, the Krull dimension and the dimension of the quasidegree set both equal the maximum of the corresponding dimensions for the composition factors in any toral filtration. □

One of the observations in [MMW05] is that hypergeometric systems $D/H_A(I,\beta)$ for varying β should be viewed as a family of D-modules fibered over $\mathbb{C}^d$. If (the holonomic rank function of the D-modules in) such a family is to behave well, it suffices to verify that it is a *holonomic family* [MMW05, Definition 2.1]. For families arising from toric modules this is done in [MMW05, Theorem 7.5], which we now generalize to the toral setting. As a matter of notation, let $b = b_1, \ldots, b_d$ be commuting variables of degree zero, so $D[b]$ is a polynomial algebra over the Weyl algebra D. For any A-graded $\mathbb{C}[\partial]$-module V, construct the *global Euler-Koszul complex* $\mathcal{K}_\bullet(E-b;V)$ of left $D[b]$-modules and *global Euler-Koszul homology* $\mathcal{H}_\bullet(E-b;V)$ by replacing D and β in Definition 2.1 with $D[b]$ and b here. Finally, if $\mathbb{C}(x)$ is the field of rational functions in $x_1, \ldots, x_n$, write $\mathcal{V}(x) = \mathbb{C}(x) \otimes_{\mathbb{C}[x]} \mathcal{V}$ for any $\mathbb{C}[x]$-module $\mathcal{V}$, including $\mathcal{V} = \mathbb{C}[b][x]$, where we set $\mathcal{V}(x) = \mathbb{C}[b](x)$.

Theorem 4.8. *If V is toral, then the sheaf $\tilde{\mathcal{V}}$ on $\mathbb{C}^d$ whose global section module is $\mathcal{V} = \mathcal{H}_0(E-b;V)$ constitutes a holonomic family over $\mathbb{C}^d$; in other words, $\mathcal{V}_\beta = \mathcal{H}_0(E-\beta;V)$ is holonomic for all $\beta \in \mathbb{C}^d$, and $\mathcal{V}(x)$ is finitely generated as a module over $\mathbb{C}[b](x)$.*

Proof. [MMW05, Proposition 7.4] holds for $I_{\rho,J}$ in place of I_A^F after harmlessly rescaling the x and ξ variables inversely to each other, which affects neither $Ax\xi$ nor the initial ideal in question. Therefore we may, in the proof of [MMW05, Theorem 7.5], simply change "toric" to "toral" and base the induction again on $I_{\rho,J}$ and $\mathbb{C}[\partial]/I_{\rho,J}$ instead of I_A^F and S_A. □

Considering b_i and β_i as elements in the polynomial ring $\mathbb{C}[b]$, we can take ordinary Koszul homology $H_\bullet(b-\beta;W)$ for any $\mathbb{C}[b]$-module W. This gets used in the generalization of [MMW05, Theorem 8.2] to arbitrary A-graded $\mathbb{C}[\partial]$-modules, which we state along with the toral generalization of [MMW05, Theorem 9.1]. For the latter, we need also the *jump arrangement* $\mathcal{Z}_{\mathrm{jump}}(V) = \bigcup_{i \leq d-1} \mathrm{qdeg}(H^i_{\mathfrak{m}}(V))$ of an A-graded module V over $\mathbb{C}[\partial]$.

Theorem 4.9. *If V is an A-graded $\mathbb{C}[\partial]$-module and $\mathcal{V} = \mathcal{H}_0(E-b;V)$, then*

$$\mathcal{H}_i(E-\beta;V) \cong H_i(b-\beta;\mathcal{V}),$$

the left and right sides being Euler-Koszul and ordinary Koszul homology, respectively. If, in addition, V is toral, then $-\beta$ lies in the jump arrangement $\mathcal{Z}_{\mathrm{jump}}(V)$ if and only if the holonomic rank of $\mathcal{H}_0(E-\beta;V)$ is not minimal (among all possible choices of β).

Proof. [MMW05, Theorem 8.2] and its proof both work verbatim for arbitrary A-graded $\mathbb{C}[\partial]$-modules. That being given, the proof of [MMW05, Theorem 9.1] works just as well for toral modules, since we have now seen that all of the earlier results in [MMW05] do. □

5. ANDEAN MODULES

The finiteness properties of toral modules encapsulated by Theorem 4.5 will be contrasted in Corollary 5.7 (the heart of which is Theorem 5.6) with the infiniteness that occurs for Andean modules. The feature of toral modules that drives the proofs in Section 4 is the toral filtration in Proposition 4.2. It would be optimal if we could simply define an Andean module, in general, to mean one that is not toral—that is, one whose Hilbert function is unbounded—and conclude a similar filtration feature for Andean modules. Alas, this notion of Andean module is too inclusive for our purposes: it does not imply a filtration characterization, in general, even though for the quotient of $\mathbb{C}[\partial]$ by a binomial primary ideal, the unbounded Hilbert function characterization is equivalent to the filtration one (Example 5.2). Therefore, we take as our foundation the filtration feature. The particular form of this feature is dictated by combinatorial primary decomposition, particularly [DMM08, Example 4.6].

Definition 5.1. An A-graded $\mathbb{C}[\partial]$-module V is *natively Andean* if there is an $\alpha \in \mathbb{Z}^d$ and an Andean quotient ring $\mathbb{C}[\partial]/I_{\rho,J}$ (Lemma 3.4) over which $V(\alpha)$ is torsion-free of rank 1 and admits a $\mathbb{Z}^J/L$-grading that refines the A-grading via $\mathbb{Z}^J/L \to \mathbb{Z}^d = \mathbb{Z}A$, where ρ is defined on $L \subseteq \mathbb{Z}^J$. If V has a finite filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_{\ell-1} \subset V_\ell = V$ whose successive quotients V_k/V_{k-1} are all natively Andean, then V is *Andean*.

Example 5.2. $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ is Andean for any Andean primary component $\mathcal{C}_{\rho,J}$ of any A-graded binomial ideal. This follows immediately from the statement of [DMM08, Example 4.6] about gradings and filtrations for primary binomial ideals.

Remark 5.3. The adjective "Andean" describes the geometry of the gradings on the $\mathbb{C}[\partial]$-modules $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$. Since $\mathcal{C}_{\rho,J}$ is a binomial ideal, there exists a commutative semigroup $\mathcal{B}$ that finely grades $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ [ES96, Proposition 1.11]. Collapsing (coarsening) the fine $\mathcal{B}$-grading to the A-grading [DMM08, Corollary 2.14] makes the $\mathcal{B}$-graded degrees sit like a high, thin mountain range over $\mathbb{Z}^d$, of unbounded elevation and supported on finitely many translates of $\mathbb{Z}A_J$. (Contrast this with the first of the equivalent conditions in Lemma 3.4.)

Here is a weak form of Euler-Koszul rigidity for Andean modules (but see Corollary 5.7).

Lemma 5.4. *If V is an Andean module and $-\beta \notin \operatorname{qdeg}(V)$, then $\mathcal{H}_i(E-\beta; W) = 0 = \mathcal{H}_i(E-\beta; V/W)$ for all i and all A-graded submodules $W \subseteq V$.*

Proof. First assume that V is natively Andean. The torsion-freeness ensures that $\operatorname{qdeg}(V)$ is a $\mathbb{Z}^d$-translate of the complex span $\mathbb{C}A_J$ of the columns of A indexed by J, so let us also assume for the moment that $\operatorname{qdeg}(V) = \mathbb{C}A_J$. The result for this V and all of its A-graded submodules follows from Lemma 4.4, because the $\mathbb{C}$-linear span of $E_1^J - \beta_1, \ldots, E_d^J - \beta_d$ contains a nonzero scalar if $\beta \notin \mathbb{C}A_J$ (some linear combination of $E_1^J, \ldots, E_d^J$ is zero, while the corresponding linear combination of $\beta_1, \ldots, \beta_d$ is nonzero, and hence a unit).

The case where V is natively Andean (or a submodule thereof) and $\operatorname{qdeg}(V) = \alpha + \mathbb{C}A_J$ is proved by applying the above argument to $V(-\alpha)$, using Lemma 2.3. The case where V is a general Andean module is proved by induction on the length of an Andean filtration, using that $\operatorname{qdeg}(V) = \operatorname{qdeg}(V') \cup \operatorname{qdeg}(V'')$ whenever $0 \to V' \to V \to V'' \to 0$ is an exact sequence. Finally, for an A-graded submodule W of a general Andean module V, intersecting W with an Andean filtration of V yields a filtration of W whose successive quotients are submodules of native Andean modules. Hence the proof of vanishing of Euler-Koszul homology by induction on the length of the filtration still applies.

The vanishing of all $\mathcal{H}_i(E-\beta; V/W)$ follows easily from the vanishing for V and for W. □

The following lemma will allow us to reduce to the case of Andean quotients $\mathbb{C}[\partial]/I_{\rho,J}$ whenever we need to work with natively Andean modules.

Lemma 5.5. *A natively Andean module V has a filtration whose successive quotients are A-graded translates of various quotients $\mathbb{C}[\partial]/I_{\rho,J}$, each being natively either toral or Andean. At least one of these quotients is natively Andean.*

Proof. By definition, V is torsion free of rank 1 over an Andean quotient $\mathbb{C}[\partial]/I_{\tau,S}$ where $S \subseteq \{1, \ldots, n\}$ and τ is a partial character (we use non-standard notation to avoid confusion with the statement we need to prove). Harmlessly rescaling the variables, we may assume that $I_{\tau,S} = I_L + \langle \partial_j : j \notin S \rangle$, so $\mathbb{C}[\partial]/I_{\tau,S}$ is a semigroup ring $\mathbb{C}[Q]$ for some $Q \subseteq \mathbb{Z}^S/L$. Replacing V with an A-graded translate, we may further assume that V is $\mathbb{Z}^S/L$-graded. Using Noetherian induction as in the proof of Lemma 2.5, we construct a filtration of V whose successive quotients are $\mathbb{Z}^S/L$-graded translates of quotients $\mathbb{C}[Q]/\mathfrak{p}_{Q'}$ modulo prime ideals $\mathfrak{p}_{Q'}$ for faces $Q' \subseteq Q$ (these are the $\mathbb{C}[\partial]/I_{\rho,J}$ of the statement). Each of these, being A-graded, is either natively toral or natively Andean. Moreover, if all of them were toral, then V would be toral as well, so the last assertion follows. □

Now we combine the Euler-Koszul theory for Andean and toral modules to conclude that the hypergeometric D-modules associated to Andean modules, if nonzero, are very large.

Theorem 5.6. *If an A-graded $\mathbb{C}[\partial]$-module V possesses a surjection to an Andean module W, and if $-\beta \in \operatorname{qdeg}(W)$, then $\mathcal{H}_0(E - \beta; V)$ has uncountably many linearly independent solutions near any general point $x \in \mathbb{C}^n$; that is, $\operatorname{Hom}_D(\mathcal{H}_0(E - \beta; V), \mathcal{O}_x)$ is a vector space of uncountable dimension over $\mathbb{C}$, where $\mathcal{O}_x$ is the local ring of analytic germs at x.*

Proof. Since a surjection of $\mathbb{C}[\partial]$-modules induces a surjection of zeroth Euler-Koszul homology $\mathcal{H}_0(E - \beta; \cdot)$, we may assume that $V = W$ is Andean.

Consider an exact sequence $0 \to V' \to V \to V'' \to 0$ of Andean modules in which V'' is natively Andean. If $-\beta \notin \operatorname{qdeg}(V'')$, then $\mathcal{H}_0(E - \beta; V') \cong \mathcal{H}_0(E - \beta; V)$ by Lemma 5.4 for V'', so we may harmlessly replace V with V'. Continuing in this manner, using induction on the length of an Andean filtration of V, we may assume that $-\beta \in \operatorname{qdeg}(V'')$. But then, since $\mathcal{H}_0(E - \beta; V)$ always surjects onto $\mathcal{H}_0(E - \beta; V'')$, we may assume that $V = V''$ is natively Andean. By Lemma 2.3, we may further assume that V is torsion-free of rank 1 over some Andean quotient $R = \mathbb{C}[\partial]/I_{\rho,J}$, and that V contains R with no A-graded translation.

Using Lemma 5.5 and its notation, take a filtration $0 = V_0 \subset V_1 \subset \cdots \subset V_\ell = V$ in which each of the successive quotients V_k/V_{k-1} is a A-graded translate of some prime quotient that is natively either toral or Andean. We are free to choose $V_1 = R$, and we do so. Let k be the largest index such that V_k/V_{k-1} is Andean and $-\beta \in \operatorname{qdeg}(V_k/V_{k-1})$, noting that such an index exists because $V_1/V_0 = R$ satisfies the condition. Since V surjects onto V/V_{k-1}, we find that $\mathcal{H}_0(E-\beta;V)$ surjects onto $\mathcal{H}_0(E-\beta;V/V_{k-1})$. Therefore, replacing V by V/V_{k-1} and $\mathbb{Z}^d$-translating again via Lemma 2.3 if necessary, it is enough to prove the case $k = 1$, with $V_1 = R$.

If the above filtration has length $\ell > 1$, then the kernel and cokernel of the homomorphism $\mathcal{H}_0(E-\beta;V_{\ell-1}) \to \mathcal{H}_0(E-\beta;V)$ are holonomic, being $\mathcal{H}_i(E-\beta;V/V_{\ell-1})$ for $i \in \{0,1\}$; this is by Theorem 4.5 if $V/V_{\ell-1}$ is toral, and by Lemma 5.4 if $V/V_{\ell-1}$ is Andean with $-\beta \notin \operatorname{qdeg}(V/V_{\ell-1})$. Therefore the desired result holds for V if and only if it holds for $V_{\ell-1}$. This argument reduces us to the case $\ell = 1$ by induction on ℓ, so we may assume that $V = R = \mathbb{C}[\partial]/I_{\rho,J}$.

The condition $-\beta \in \operatorname{qdeg}(R)$ means exactly that $-\beta$, or equivalently β, lies in the complex column span $\mathbb{C}A_J$. Let $\hat{A}$ be a matrix for the projection $\mathbb{Z}^J \to \mathbb{Z}^J/L$, and write $\mathbb{Z}\hat{A} = \mathbb{Z}^J/L$. If $\hat{\beta}$ is a vector in $\mathbb{C}\hat{A}$ mapping to β under the surjection to $\mathbb{C}A_J$ afforded by Lemma 3.4, then denote by $\hat{E} - \hat{\beta}$ the sequence of Euler operators associated to $\hat{A}$. Thought of as elements in the space of affine linear functions $\mathbb{Z}^J \to \mathbb{C}$, the Euler operators $E_1^J - \beta_1, \ldots, E_d^J - \beta_d$ truncated from $E - \beta$ generate a sublattice $\mathbb{Z}\{E^J - \beta\}$ properly contained in the sublattice $\mathbb{Z}\{\hat{E} - \hat{\beta}\}$ generated by $\hat{E} - \hat{\beta}$. The binomial hypergeometric system $D/H_{\hat{A}}(I_{\rho,J}, \hat{\beta})$ is holonomic of positive rank by Theorem 4.5 (for $\mathbb{Z}^J/L$-graded toral $\mathbb{C}[\partial_J]$-modules, via Lemma 4.4). Its solutions are also solutions of $D/H_A(I_{\rho,J}, \beta)$ because

$$H_A(I_{\rho,J}, \beta) = D \cdot \langle I_{\rho,J}, \mathbb{Z}\{E^J - \beta\}\rangle \subseteq D \cdot \langle I_{\rho,J}, \mathbb{Z}\{\hat{E} - \hat{\beta}\}\rangle = H_{\hat{A}}(I_{\rho,J}, \hat{\beta}).$$

On the other hand, for any pair of distinct lifts $\hat{\beta} \neq \hat{\beta}'$, the linear span of $\mathbb{Z}\{\hat{E} - \hat{\beta}\}$ together with $\mathbb{Z}\{\hat{E} - \hat{\beta}'\}$ contains a nonzero scalar. It follows that the solutions to $D/H_{\hat{A}}(I_{\rho,J}, \hat{\beta})$ for varying $\hat{\beta}$ are linearly independent. The direct sum of these (local) solution spaces is therefore an uncountable-dimensional subspace of the (local) solutions to $\mathcal{H}_0(E-\beta;R) = D/H_A(I_{\rho,J}, \beta)$. □

Summarizing the above results, let us emphasize the dichotomy between toral and Andean modules by recording the Andean analogue of Theorem 4.5.

Corollary 5.7. *The following are equivalent for an Andean $\mathbb{C}[\partial]$-module V and $\beta \in \mathbb{C}^d$.*

0. $\mathcal{H}_0(E-\beta;V)$ has countable-dimensional local solution space.
1. $\mathcal{H}_0(E-\beta;V)$ has finite-dimensional local solution space.
2. $\mathcal{H}_0(E-\beta;V) = 0$.
3. $\mathcal{H}_i(E-\beta;V) = 0$ for all $i \geq 0$.
4. $-\beta \notin \operatorname{qdeg}(V)$. □

6. BINOMIAL D-MODULES

Using the functoriality of Euler-Koszul homology, we now deduce the holonomicity, regularity, and other structural properties of arbitrary binomial D-modules, including the binomial Horn systems which motivated and presaged the developments here. Our first principal result is the specification, for any A-graded binomial ideal I, of an arrangement of finitely many affine subspaces of $\mathbb{C}^d$ such that the binomial D-module $D/H_A(I,\beta)$ is holonomic precisely when $-\beta$ lies outside of it (Theorem 6.3). Moreover, holonomicity occurs if and only if the vector space of local solutions to $H_A(I,\beta)$ has finite dimension. The subspace arrangement arises from the primary decomposition of I into its toral and Andean components. When $D/H_A(I,\beta)$ is holonomic, it is also regular holonomic if and only if I is $\mathbb{Z}$-graded in the standard sense. Finally, we construct another finite affine subspace arrangement in $\mathbb{C}^d$ such that for $-\beta$ outside of it, the binomial D-module splits as a direct sum of primary toral binomial D-modules (Theorem 6.8).

For the duration of this section, fix an A-graded binomial ideal $I \subseteq \mathbb{C}[\partial_1,\dots,\partial_n]$ and fix an irredundant primary decomposition as in Theorem 3.2. Thus, as in Lemma 3.4, some of the quotients $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ are toral and some are Andean. Much of what we do is independent of the particular primary decomposition, since the data we typically need come from the quasidegrees of certain related modules. For example, the holonomicity in Theorem 6.3 is clearly independent of the primary decomposition.

Definition 6.1. The *Andean arrangement* $\mathcal{Z}_{\text{Andean}}(I)$ is the union of the quasidegree sets $\operatorname{qdeg}(\mathbb{C}[\partial]/\mathcal{C}_{\rho,J})$ for the Andean primary components $\mathcal{C}_{\rho,J}$ of I.

Lemma 6.2. *The Andean arrangement $\mathcal{Z}_{\text{Andean}}(I)$ is a union of finitely many integer translates of the subspaces $\mathbb{C}A_J \subseteq \mathbb{C}^n$ for which there is an Andean associated prime $I_{\rho,J}$.*

Proof. Apply Lemma 2.5 to an Andean filtration of each Andean component $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$. □

Theorem 6.3. *Given the A-graded binomial ideal $I \subseteq \mathbb{C}[\partial]$, the following are equivalent.*

0. *The vector space of local solutions to $H_A(I,\beta)$ has countable dimension.*
1. *The vector space of local solutions to $H_A(I,\beta)$ has finite dimension.*
2. *The binomial D-module $D/H_A(I,\beta)$ is holonomic.*
3. *The Euler-Koszul homology $\mathcal{H}_i(E-\beta;\mathbb{C}[\partial]/I)$ is holonomic for all i.*
4. $-\beta \notin \mathcal{Z}_{\text{Andean}}(I)$.

For I standard $\mathbb{Z}$-graded, these are equivalent to regular holonomicity of $\mathcal{H}_i(E-\beta;\mathbb{C}[\partial]/I)$. Moreover, the existence of a parameter β for which $\mathcal{H}_0(E-\beta;\mathbb{C}[\partial]/I)$ is regular holonomic implies that I is $\mathbb{Z}$-graded.

Proof. The last claim follows from the rest by Theorem 4.5 and results in [Hot91, SW08]. Item 1 trivially implies item 0. Item 2 implies item 1 because holonomic systems have finite rank. Item 3 implies item 2 by Definition 1.6 and Example 2.2. If $-\beta \in \mathcal{Z}_{\text{Andean}}(I)$, then $-\beta \in \operatorname{qdeg}(\mathbb{C}[\partial]/\mathcal{C}_{\rho,J})$ for some Andean component $\mathcal{C}_{\rho,J}$, so item 0 implies item 4 by Theorem 5.6 for the surjection $\mathbb{C}[\partial]/I \twoheadrightarrow \mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$. Finally, item 4 implies item 3 by

Theorem 4.5 and Proposition 6.4, below, given that $\mathbb{C}[\partial]/\bigcap_{I_{\rho,J}\text{ toral}} \mathcal{C}_{\rho,J}$ is a submodule of $\bigoplus_{I_{\rho,J}\text{ toral}} \mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ and is hence toral. □

Proposition 6.4. *Let $I_{\text{toral}} = \bigcap_{I_{\rho,J}\text{ toral}} \mathcal{C}_{\rho,J}$ be the intersection of the toral primary components of I. If $-\beta$ lies outside of the Andean arrangement of I, then the natural surjection $\mathbb{C}[\partial]/I \twoheadrightarrow \mathbb{C}[\partial]/I_{\text{toral}}$ induces an isomorphism in Euler-Koszul homology:*

$$\mathcal{H}_i(E-\beta;\mathbb{C}[\partial]/I) \cong \mathcal{H}_i(E-\beta;\mathbb{C}[\partial]/I_{\text{toral}}) \text{ for all } i \text{ when } -\beta \notin \mathcal{Z}_{\text{Andean}}(I).$$

Proof. If I_{Andean} is the intersection of the Andean primary components of I, then

$$\frac{I_{\text{toral}}}{I} = \frac{I_{\text{toral}}}{I_{\text{toral}} \cap I_{\text{Andean}}} \cong \frac{I_{\text{toral}} + I_{\text{Andean}}}{I_{\text{Andean}}}$$

is a submodule of $\mathbb{C}[\partial]/I_{\text{Andean}}$, which in turn is a submodule of $\bigoplus_{I_{\rho,J}\text{ Andean}} \mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$. Since $\mathcal{Z}_{\text{Andean}}(I)$ is the quasidegree set of this Andean direct sum, the exact sequence

$$0 \to \frac{I_{\text{toral}}}{I} \to \frac{\mathbb{C}[\partial]}{I} \to \frac{\mathbb{C}[\partial]}{I_{\text{toral}}} \to 0$$

yields isomorphisms $\mathcal{H}_i(E-\beta;\mathbb{C}[\partial]/I) \cong \mathcal{H}_i(E-\beta;\mathbb{C}[\partial]/I_{\text{toral}})$ of Euler-Koszul homology for all i, by Lemma 5.4 for I_{toral}/I. □

Now we move on to the question of when $D/H_A(I,\beta)$ splits into a direct sum.

Definition 6.5. The *primary cokernel module* P_I is defined by the exact sequence

$$0 \to \frac{\mathbb{C}[\partial]}{I} \to \bigoplus_{I_{\rho,J}\in \operatorname{Ass}(I)} \frac{\mathbb{C}[\partial]}{\mathcal{C}_{\rho,J}} \to P_I \to 0.$$

The *primary arrangement* is $\mathcal{Z}_{\text{primary}}(I) = \operatorname{qdeg}(P_I) \cup \mathcal{Z}_{\text{Andean}}(I)$.

Proposition 6.6. *The primary arrangement $\mathcal{Z}_{\text{primary}}(I)$ is a union of finitely many integer translates of subspaces $\mathbb{C}A_J \subseteq \mathbb{C}^n$. If there exists $\beta \in \mathbb{C}^d$ such that the local solution space of $H_A(I,\beta)$ has finite dimension, then $\mathcal{Z}_{\text{primary}}(I)$ is a proper Zariski-closed subset of $\mathbb{C}^d$.*

Proof. The first sentence is by Lemma 2.5. For the second sentence, let $(P_I)_{\text{toral}}$ be the image in P_I of the direct sum $\bigoplus_{\text{toral}} \mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$. A point in $\operatorname{qdeg}(P_I)$ that does not lie in $\mathcal{Z}_{\text{Andean}}(I)$ must necessarily be a quasidegree of $(P_I)_{\text{toral}}$; that is

$$\mathcal{Z}_{\text{primary}}(I) = \operatorname{qdeg}\big((P_I)_{\text{toral}}\big) \cup \mathcal{Z}_{\text{Andean}}(I). \tag{6.1}$$

The existence of our β immediately implies that $\mathcal{Z}_{\text{Andean}}(I)$ is a proper Zariski-closed subset of $\mathbb{C}^n$, so by (6.1) we need only prove the same thing for $\operatorname{qdeg}((P_I)_{\text{toral}})$. The module $(P_I)_{\text{toral}}$ is supported on the union of the toric subvarieties $T_{\rho,J} = \operatorname{Spec}(\mathbb{C}[\partial]/I_{\rho,J})$ for the toral associated primes of I; this much is by definition. However, the map $\mathbb{C}[\partial]/I \to \bigoplus_{\operatorname{Ass}(I)} \mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ is an isomorphism locally at a point x whenever x lies in only one of the associated varieties $T_{\rho,J}$ (toral or otherwise). Therefore $(P_I)_{\text{toral}}$ is supported on the union of the pairwise intersections of the toral toric varieties $T_{\rho,J}$ associated to I. Hence it is enough to show that if R is the coordinate ring of the intersection $T_{\rho,J} \cap T_{\rho',J'}$ of any two distinct toral varieties, then $\operatorname{qdeg}(R)$ is a proper Zariski-closed subset of $\mathbb{C}^d$. This is a consequence of Lemma 4.7. □

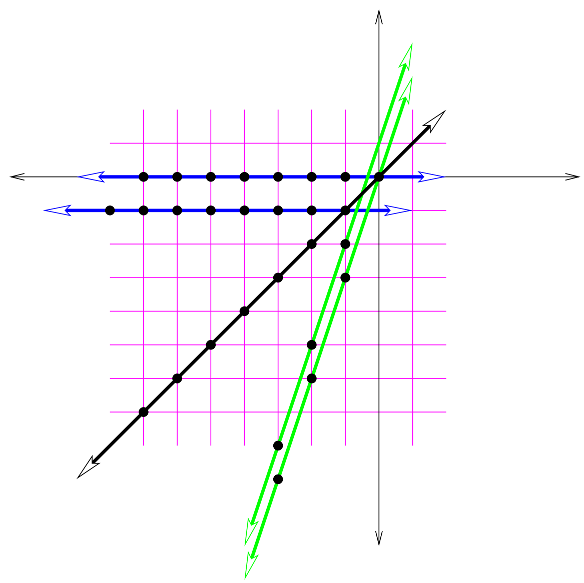

FIGURE 1. Primary arrangement $\mathcal{Z}_{\text{primary}}(I)$ of the binomial ideal I in Example 6.7

Example 6.7. In Examples 2.6, 3.1, and 3.5, the primary arrangement $\mathcal{Z}_{\text{primary}}(I)$ consists of the five bold lines in Figure 1. The diagonal line through $-\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ is the Andean arrangement $\mathcal{Z}_{\text{Andean}}(I)$ by Examples 2.6 and 3.1. On the other hand, the pairwise intersections of the toral components of I all equal $\langle bc, ad, b^2, ac, c^2, bd, b-e\rangle$, which has primary decomposition

$$\langle bc, ad, b^2, ac, c^2, bd, b-e\rangle = \langle b^2, c, d, b-e\rangle \cap \langle a, b, c^2, b-e\rangle.$$

The set of true degrees of P_I that lie outside of $\mathcal{Z}_{\text{Andean}}(I)$ coincides with the true degree set $\operatorname{tdeg}(\mathbb{C}[a,b,c,d,e]/\langle bc, ad, b^2, ac, c^2, bd, b-e\rangle)$, which consists simply of the A-degrees of the monomials in a, b, c, and d that are nonzero in this quotient. The exponent vectors of these monomials are those of the form

$$\begin{bmatrix}\alpha\\0\\0\\0\end{bmatrix}, \begin{bmatrix}\alpha\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\\delta\end{bmatrix}, \text{ or } \begin{bmatrix}0\\0\\1\\\delta\end{bmatrix}$$

for $\alpha \in \mathbb{N}$ and $\delta \in \mathbb{N}$, so $\operatorname{tdeg}(P_I) \smallsetminus \mathcal{Z}_{\text{Andean}}(I)$ consists of the lattice points having the form

$$\left[\begin{smallmatrix}-\alpha\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}-\alpha-1\\-1\end{smallmatrix}\right], \left[\begin{smallmatrix}-\delta\\-3\delta\end{smallmatrix}\right], \text{ or } \left[\begin{smallmatrix}-\delta-1\\-3\delta-2\end{smallmatrix}\right],$$

keeping in mind that the degrees of the variables are the negatives of the columns of A. These true degrees are plotted as black dots in Figure 1. The pair of horizontal lines comes from $\langle b^2, c, d, b-e\rangle$, while the pair of steep diagonal lines comes from $\langle b, c^2, d, b-e\rangle$.

Theorem 6.8. *Assume that $-\beta$ lies outside of the primary arrangement $\mathcal{Z}_{\text{primary}}(I)$. Then*

$$\mathcal{H}_i(E-\beta; \mathbb{C}[\partial]/I) \cong \bigoplus_{I_{\rho,J} \text{ toral}} \mathcal{H}_i(E-\beta; \mathbb{C}[\partial]/\mathcal{C}_{\rho,J})$$

for all i, the sum being over all toral associated primes of I from Theorem 3.2. In particular,

$$D/H_A(I,\beta) \cong \bigoplus_{I_{\rho,J} \text{ toral}} D/H_A(\mathcal{C}_{\rho,J}, \beta).$$

Proof. Assume that $-\beta \notin \mathcal{Z}_{\text{primary}}(I)$. Resuming the notation from the proof of Proposition 6.6, we have an exact sequence $0 \to (P_I)_{\text{toral}} \to P_I \to P_I/(P_I)_{\text{toral}} \to 0$. The direct sum $\bigoplus_{\text{Andean}} \mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ over the Andean components of I surjects onto $P_I/(P_I)_{\text{toral}}$. Hence, by Lemma 5.4, we deduce that $\mathcal{H}_i(E-\beta; P_I/(P_I)_{\text{toral}}) = 0$ for all i. Consequently, $\mathcal{H}_i(E-\beta; P_I) \cong \mathcal{H}_i(E-\beta; (P_I)_{\text{toral}})$ for all i. But the latter is zero for all i by Theorem 4.5 because $-\beta \notin \text{qdeg}(P_I) \supseteq \text{qdeg}((P_I)_{\text{toral}})$. Therefore, applying Euler-Koszul homology to the exact sequence in Definition 6.5, and using Lemma 5.4 to note that this kills the Andean summands, we have proved the first display. The second is simply the $i = 0$ case. □

Here is our final arrangement, outside of which the holonomic rank of $H_A(I,\beta)$ is minimal.

Definition 6.9. Given an A-graded binomial I, the *jump arrangement* of I is the union

$$\mathcal{Z}_{\text{jump}}(I) = \mathcal{Z}_{\text{Andean}}(I) \cup \bigcup_{i=0}^{d-1} \text{qdeg}\big(H^i_{\mathfrak{m}}(\mathbb{C}[\partial]/I_{\text{toral}})\big)$$

of the Andean arrangement of I with the quasidegrees of the local cohomology of $\mathbb{C}[\partial]/I_{\text{toral}}$ in cohomological degrees at most $d-1$.

Once the holonomic rank of a binomial D-module is minimal, we can quantify it exactly. Let $\mu_{\rho,J}$ be multiplicity of $I_{\rho,J}$ in I (or equivalently, in the primary component $\mathcal{C}_{\rho,J}$ of I). Denote by $\text{vol}(A_J)$ the volume of the convex hull of A_J and the origin, normalized so that a lattice simplex in the group $\mathbb{Z}A_J$ generated by the columns of A_J has volume 1.

Theorem 6.10. *If $\mathcal{Z}_{\text{Andean}}(I) \neq \mathbb{C}^d$, then $H_A(I,\beta)$ has minimal rank at β if and only if $-\beta$ lies outside of the jump arrangement $\mathcal{Z}_{\text{jump}}(I)$, and this minimal rank is*

$$\text{rank}\big(D/H_A(I,\beta)\big) = \sum_{I_{\rho,J} \text{ toral of dim. } d} \mu_{\rho,J} \cdot \text{vol}(A_J).$$

Proof. Assume that $\mathcal{Z}_{\text{Andean}}(I) \neq \mathbb{C}^d$, and denote by X the complement of $-\mathcal{Z}_{\text{Andean}}(I)$ in $\mathbb{C}^d$. The global Euler-Koszul homology $\mathcal{H}_0(E-b; \mathbb{C}[\partial]/I)$ determines a sheaf on $\mathbb{C}^d$, and hence a sheaf $\mathcal{F}$ on X by restriction. We claim that $\mathcal{F}$ is a holonomic family [MMW05, Definition 2.1] over X. In fact, we claim that $\mathcal{F}$ is the restriction to X of the family determined by $\mathcal{H}_0(E-b; \mathbb{C}[\partial]/I_{\text{toral}})$, which is a holonomic family on all of $\mathbb{C}^d$ by Theorem 4.8. Our claim is immediate from the sheaf (i.e., global Euler-Koszul) version Proposition 6.4, which says that for all i, if $\beta \in X$ then $\mathcal{H}_i(E-b; \mathbb{C}[\partial]/I) \cong \mathcal{H}_i(E-b; \mathbb{C}[\partial]/I_{\text{toral}})$ in a neighborhood of β. This follows by the same proof as Proposition 6.4 itself, given the global version of Lemma 5.4. This global version, in turn, follows from the same proof as Lemma 5.4 itself with β_i replaced by b_i for all i, the point being that $b_i = (b_i - \beta_i) + \beta_i$ is a unit locally in $\mathbb{C}^d$ near β, since $b_i - \beta_i$ lies in the maximal ideal at β.

The statement about minimality of rank is now a consequence of Theorem 4.9 for $V = \mathbb{C}[\partial]/I_{\text{toral}}$, noting that the rank is infinite for $\beta \notin X$ by Theorem 6.3. To compute this minimal rank, we may assume that β is as generic as we like. In particular, we assume that $-\beta$ lies outside of the primary arrangement, and also (by Lemma 4.7) outside of $\text{qdeg}(\mathbb{C}[\partial]/\mathcal{C}_{\rho,J})$ for the components of dimension less than d. Using Theorem 6.8, we will be done once we show that $H_A(\mathcal{C}_{\rho,J},\beta)$ has rank $\mu_{\rho,J} \cdot \text{vol}(A_J)$ for generic β.

To do this, take a toral filtration of $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$. We are guaranteed that the number of successive quotients of dimension d is precisely the multiplicity of $I_{\rho,J}$ in $\mathcal{C}_{\rho,J}$, and that all of the dimension d successive quotients are actually $\mathbb{Z}^d$-translates of $\mathbb{C}[\partial]/I_{\rho,J}$ itself. Therefore, choosing β to miss the quasidegree sets of the other successive quotients, we find that the rank of $H_A(\mathcal{C}_{\rho,J}, \beta)$ equals the multiplicity $\mu_{\rho,J}$ times the generic rank of $H_A(I_{\rho,J}, \beta) = H_{A_J}(I_{\rho,J}, \beta)$, which is $\mathrm{vol}(A_J)$ by [Ado94]. □

Remark 6.11. If $I = I(B)$ is a lattice basis ideal (Example 3.3), then the sum in Theorem 6.10 can be simplified by gathering the terms $\mu_{\rho,J} \cdot \mathrm{vol}(A_J)$ for which the domain of ρ is a fixed toral saturated sublattice $L \subseteq \mathbb{Z}^J$. The single term that results is $\mu(L, J) \cdot \mathrm{vol}(A_J) = |L/\mathbb{Z}B \cap \mathbb{Z}^J| \cdot \mu_{\rho,J} \cdot \mathrm{vol}(A_J)$, where $\rho : L \to \mathbb{C}^*$ is any partial character that is trivial on $\mathbb{Z}B$. Indeed, the number of choices for ρ is $|L/\mathbb{Z}B \cap \mathbb{Z}^J|$, and once $I_{\rho,J}$ is associated to $I(B)$, the same is true for any other choice of ρ; this is because rescaling the variables by a partial character that is trivial on $\mathbb{Z}B$ induces an automorphism of the polynomial ring fixing the lattice basis ideal $I(B)$. For the same reason, the multiplicities of the various choices of $I_{\rho,J}$ in $I(B)$ are all equal.

Remark 6.12. The arrangement that we should require $-\beta$ to avoid for β to be called truly *generic* is the union of the jump arrangement $\mathcal{Z}_{\text{jump}}(I)$ and the *top arrangement* $\mathcal{Z}_{\text{top}}(I) = \mathrm{qdeg}\big(\bigoplus_{\text{toral}<d} \mathbb{C}[\partial]/\mathcal{C}_{\rho,J}\big)$, where the direct sum is over all toral components of I with $\dim(\mathbb{C}[\partial]/I_{\rho,J}) \leq d-1$. For $-\beta \notin \mathcal{Z}_{\text{jump}}(I) \cup \mathcal{Z}_{\text{top}}(I)$, the module $D/H_A(I, \beta)$ has minimal holonomic rank and decomposes as a direct sum over the dimension d toral components.

Corollary 6.13. *If I is standard $\mathbb{Z}$-graded without any Andean components, and $\mathbb{C}[\partial]/I$ has Krull dimension d, then the generic rank of $H_A(I, \beta)$ equals the $\mathbb{Z}$-graded degree of I.* □

We close this section by illustrating a particular case of a Mellin system [Mel21, DS07]. Such systems arise when showing that algebraic functions satisfy hypergeometric equations. The goal of the example is to give an instance when the local solution space of the binomial D-module $D/H_A(I, \beta)$ for some nonzero parameter β fails to split as a direct sum of the local solution spaces to binomial D-modules arising from components. Note that $\beta = 0$ always lies in the primary arrangement: the residue field $\mathbb{C} = \mathbb{C}[\partial]/\mathfrak{m}$ is a quotient of every primary component $\mathbb{C}[\partial]/\mathcal{C}_{\rho,J}$ because the A-grading is positive (i.e., $\mathbb{N}A$ is a pointed semigroup).

Example 6.14. Let

$$A = \left[\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 0 \end{smallmatrix}\right] \quad \text{and} \quad B^t = \left[\begin{smallmatrix} -2 & 3 & 0 & -1 \\ -1 & 0 & 3 & -2 \end{smallmatrix}\right].$$

In this case we have

$$I_{\mathbb{Z}B} = I(B) = \langle \partial_1^2\partial_4 - \partial_2^3, \partial_1\partial_4^2 - \partial_3^3 \rangle \subseteq \mathbb{C}[\partial_1, \partial_2, \partial_3, \partial_4].$$

That is, the lattice basis ideal $I(B)$ coincides with the lattice ideal $I_{\mathbb{Z}B}$. The primary decomposition of $I_{\mathbb{Z}B}$ is obtained from that of the ideal I in Examples 2.6 and 6.7 by omitting the Andean component $\langle a, c, d\rangle$ and erasing all occurrences of $b - e$. Thus the primary arrangement of $I_{\mathbb{Z}B}$ consists of the four lines in Figure 1 corresponding to toral components.

Let $\beta = -\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. The solutions of the system $H_A\big(I_{\mathbb{Z}B}, -\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\big)$ are as follows. For $x = (x_1, x_2)$, let $z_1(t)$, $z_2(t)$ and $z_3(t)$ be the local roots in a neighborhood of $(0, 0)$ of

$$z^3 + x_1z^2 + x_2z + 1 = 0.$$

By [Stu00], a local basis of solutions of the A-hypergeometric system $H_A\big(-\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\big) = H_A\big(I_A, -\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\big)$ for the toric ideal I_A (1.2) is given by the three roots of the homogeneous equation

$$x_0 z^3 + x_1 z^2 + x_2 z + x_3 = 0,$$

and the solutions for the other two components are the roots of

$$x_0 z^3 + x_1 z^2 + \omega x_2 z + x_3 = 0 \quad \text{and} \quad x_0 z^3 + x_1 z^2 + \omega^2 x_2 z + x_3 = 0,$$

where ω is a primitive cube root of 1. The system $H_A\big(I_{\mathbb{Z}B}, -\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\big)$ has nine algebraic solutions coming from the roots $z = z(x_0, x_1, x_2, x_3)$ of the above equations.

This looks good: the quotient $\mathbb{C}[\partial]/I_{\mathbb{Z}B}$ is Cohen-Macaulay, so $\mathcal{H}_0(E - \beta, \mathbb{C}[\partial]/I_{\mathbb{Z}B})$ has holonomic rank that is constant as a function of $\beta \in \mathbb{C}^2$, by the rank minimality in Theorem 6.10, and equal to 9 because $\operatorname{vol}(A) = 3$.

However, the nine algebraic solutions mentioned above only span a vector space of dimension 7, not 9. This means that there are two extra linearly independent local solutions, which are non-algebraic; see [DS07, Example 4.2, Theorem 4.3, Example 4.4].

The binomial D-module explanation for this collapsing from dimension 9 to dimension 7, and the concomitant extra two logarithmic solutions, is that $-\beta = \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] \in \mathcal{Z}_{\text{primary}}(I_{\mathbb{Z}B})$; again see Figure 1. Let us be more precise. The exact sequence in Definition 6.5 reads

$$0 \to \mathbb{C}[\partial]/I_{\mathbb{Z}B} \to R_0 \oplus R_1 \oplus R_2 \to P_{I_{\mathbb{Z}B}} \to 0,$$

where $R_i = \mathbb{C}[\partial_1, \partial_2, \partial_3, \partial_4]/\langle \omega^i \partial_2\partial_3 - \partial_1\partial_4, \partial_2^2 - \omega^i \partial_1\partial_3, \omega^{2i}\partial_3^2 - \partial_2\partial_4\rangle$. The surjection to $P_{I_{\mathbb{Z}B}}$ factors through the projection $R_0 \oplus R_1 \oplus R_2 \to \overline{R} \oplus \overline{R} \oplus \overline{R}$, where $\overline{R}$ is the monomial quotient $\mathbb{C}[\partial]/\langle \partial_2\partial_3, \partial_1\partial_4, \partial_2^2, \partial_1\partial_3, \partial_3^2, \partial_2\partial_4\rangle$, the coordinate ring of the intersection scheme of any pair of irreducible components of the variety of $I_{\mathbb{Z}B}$. The image of $\mathbb{C}[\partial]/I_{\mathbb{Z}B}$ in this projection is the diagonal copy of $\overline{R}$, so $P_{I_{\mathbb{Z}B}}$ is a direct sum $\overline{R} \oplus \overline{R}$ of two copies of $\overline{R}$.

On the other hand, each of the rings R_i is also Cohen-Macaulay, so the only nonvanishing Euler-Koszul homology of $R_0 \oplus R_1 \oplus R_2$ is the zeroth. Thus we have an exact sequence

$$0 \to \mathcal{H}_1(E - \beta; P_{I_{\mathbb{Z}B}}) \to D/H_A(I_{\mathbb{Z}B}, \beta) \to \bigoplus_{i=0}^{2} \mathcal{H}_0(E - \beta; R_i) \to \mathcal{H}_0(E - \beta; P_{I_{\mathbb{Z}B}}) \to 0.$$

In general, for $-\beta$ lying on precisely one of the four lines in $\mathcal{Z}_{\text{primary}}(I_{\mathbb{Z}B}) = \operatorname{qdeg}(P_{I_{\mathbb{Z}B}})$, the leftmost and rightmost D-modules here have rank precisely 2, and this is the 2 that causes the dimension collapse and the pair of logarithmic solutions to appear.

Given our choice of parameter $\beta = -\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, for instance, the $\mathcal{H}_1$ and the $\mathcal{H}_0$ in question are isomorphic to one another, since both are isomorphic to a direct sum of two copies of $\mathcal{H}_0(E - \big(-\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\big); \partial_3\mathbb{C}[\partial]/\langle \partial_1, \partial_2, \partial_3\rangle)$, where the ∂_3 in front of $\mathbb{C}[\partial]$ means to take an appropriate A-graded translate (namely by $\deg(\partial_2) = -\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$); this corresponds to the upper of the two steep diagonal lines in Figure 1.

7. Local solutions of Horn D-modules

We now return to the Horn hypergeometric D-modules—that is, binomial D-modules arising from lattice basis ideals—that motivated this work. Theorem 7.14, the main result of this section, provides a combinatorial formula for the generic rank of a binomial Horn system by explicitly describing a basis for its local solution space. The basis we construct involves GKZ hypergeometric functions.

Throughout this section, let B and A be integer matrices as in Convention 1.4. Since we have an explicit description for the components of a lattice basis ideal $I(B)$ at toral minimal primes, namely Example 3.7, we make use of it to compute—just as explicitly—the local solutions for generic $\beta \in \mathbb{C}^d$ of the corresponding hypergeometric system.

Convention 7.1. Suppose that after permuting the rows and columns of B, there results a decomposition of B as in (3.1), where M is a $q \times p$ matrix of full rank q. Write $\bar{J} = \bar{J}(M)$ for the q rows occupied by M inside of B (before permuting), and let $J = \{1, \dots, n\} \smallsetminus \bar{J}$ be the rows occupied by B_J. Split the variables $x_1, \dots, x_n$ and $\partial_1, \dots, \partial_n$ into two blocks each:

$$x_J = \{x_j : j \in J\} \quad \text{and} \quad x_{\bar{J}} = \{x_j : j \notin J\}.$$
$$\partial_J = \{\partial_j : j \in J\} \quad \text{and} \quad \partial_{\bar{J}} = \{\partial_j : j \notin J\}.$$

As before, A_J is the submatrix of A with columns $\{a_j : j \in J\}$.

With the notation above, fix for the remainder of this article a toral prime $I_{\rho,J}$ of $I(B)$. Since $I(B)$ is generated by $m = n - d$ elements, $I_{\rho,J}$ has dimension at least d. On the other hand, toral primes can have dimension at most d, by Lemma 4.7. Thus we have the following.

Lemma 7.2. *All toral primes of the lattice basis ideal $I(B)$ have dimension exactly d and are minimal primes of $I(B)$.* □

Observation 7.3. Since the dimension of $I_{\rho,J}$ equals $n - q - (m - p) = d + p - q$, if $I_{\rho,J}$ is toral, then the previous lemma implies that $q = p$. Thus, from now on, the matrix M is a $q \times q$ mixed invertible matrix (and q is allowed to be 0).

Recall from Example 3.7 that for a toral minimal prime $I_{\rho,J}$, the component can be written as $\mathcal{C}_{\rho,J} = I(B) + I_\rho + U_M$, where $U_M \subseteq \mathbb{C}[\partial_{\bar{J}}]$ is the ideal $\mathbb{C}$-linearly spanned by all monomials with exponent vectors in the union of the unbounded M-subgraphs of $\mathbb{N}^{\bar{J}}$ (Definition 3.6).

We will see in Theorem 7.14 that in order to construct local solutions of $H_A(\mathcal{C}_{\rho,J}, \beta)$ we need two ingredients: local solutions of a GKZ-type system $H_{A_J}(I_\rho, \beta')$ ("coming" from the binomials in $\mathcal{C}_{\rho,J}$) and polynomial solutions of the constant coefficient system $I(M) = \langle \partial^u - \partial^v : u - v \text{ is a column of } M,\ u, v \in \mathbb{N}^n \rangle$ ("coming" from the monomials in $\mathbb{C}_{\rho,J}$).

In order to describe the solutions (7.1) below, we start by considering the solutions of $I(M)$. As it turns out, solving the differential equations $I(M)$ is equivalent to finding the M-subgraphs of $\mathbb{N}^{\bar{J}}$.

Lemma 7.4. *Let M be a $q \times q$ mixed invertible integer matrix, and assume that $q > 0$. Fix $\gamma \in \mathbb{N}^{\bar{J}}$, and denote by Γ the M-subgraph containing γ.*

1. *The system $I(M)$ of differential equations has a unique formal power series solution of the form $G_\gamma = \sum_{u\in\Gamma} \lambda_u x^u$ in which $\lambda_\gamma = 1$.*
2. *The other coefficients λ_u of G_γ for $u \in \Gamma$ are all nonzero.*

This lemma will be proved together with Proposition 7.6.

Notation 7.5. Given a $q\times q$ mixed invertible matrix M, we fix a set $\mathcal{S}(M) \subset \mathbb{N}^{\bar{J}}$ of representatives for the bounded M-sugraphs of $\mathbb{N}^{\bar{J}}$. In particular, the cardinality of $\mathcal{S}(M)$ equals the number of bounded M-subgraphs, which we denote by μ_M. If $q = 0$, we set $\mathcal{S}(M) = \{\varnothing\}$ and declare μ_M to be 1.

Proposition 7.6. *With the notation from Lemma 7.4 and Notation 7.5,*

1. *The set $\{G_\gamma : \gamma$ runs over a set of representatives for the M-subgraphs of $\mathbb{N}^{\bar{J}}\}$ is a basis for the space of all formal power series solutions of $I(M)$.*
2. *The set $\{G_\gamma : \gamma \in \mathcal{S}(M)\}$ is a basis for the space of polynomial solutions of $I(M)$.*

Proof of Lemma 7.4 and Proposition 7.6. We begin with the first statement from Lemma 7.4. If $\Gamma = \{\gamma\}$ then $G_\gamma = x^\gamma$. We check that this is a solution of $I(M)$ working by contradiction. Let w be a column of M such that $\partial^{w_+}x^\gamma \neq \partial^{w_-}x^\gamma$. Then one of these terms is nonzero, say $\partial^{w_+}x^\gamma$, so that $\gamma - w_+ \in \mathbb{N}^{\bar{J}}$. But then $\gamma - w_+ + w_- = \gamma - w \in \mathbb{N}^{\bar{J}}$, and so $\gamma - w \in \Gamma$, a contradiction, because $\gamma - w \neq \gamma$ and Γ is a singleton.

Now assume that Γ is not a singleton, and fix $u \in \Gamma$ such that $u-\gamma = w$ is a column of M. We want to define the coefficients of G_γ, and we will start with λ_u. Since $u-\gamma = w = w_+ - w_-$, we have $u - w_+ = \gamma - w_- \in \mathbb{N}^{\bar{J}}$, since u and γ both lie in $\mathbb{N}^{\bar{J}}$ and the supports of w_+ and w_- are disjoint. Set $\lambda_u = \partial^{w_-}(x^\gamma)/\partial^{w_+}(x^u)$, and observe that numerator and denominator are *nonzero* constant multiples of $x^{u-w_+} = x^{\gamma-w_-}$. Use this procedure to define the coefficients corresponding to the neighbors of γ. Now, if we know λ_u and we are given a neighbor $u' \in \Gamma$ of u, say $u' - u = w$, then set $\lambda_{u'} = \partial^{w_-}(x^u)/\lambda_u\partial^{w_+}(x^{u'})$. Propagating this procedure along Γ we obtain all of the coefficients λ_u. The formal power series G_γ defined this way is tailor-made to be a solution of $I(M)$.

Since M-subgraphs are disjoint, it is clear that the series G_γ are linearly independent. Now let $G = \sum_{u\in\mathbb{N}^{\bar{J}}} \nu_u x^u$ be a formal power series solution of $I(M)$. We claim that $G - \nu_\gamma G_\gamma$ has coefficient zero on all monomials from Γ. This follows from the fact that $G - \nu_\gamma G_\gamma$ has coefficient zero on the monomial x^γ; indeed, if the difference contained a monomial from Γ, it would have to contain x^γ with a nonzero coefficient, as can be seen by the propagation argument from before. (The uniqueness of G_γ that we need for Lemma 7.4 also follows from this argument.) It is now clear that our candidate power series solution basis is a spanning set, and the statement for polynomial solutions has the same proof. ☐

Remark 7.7. Indeed, for any system of constant coefficient differential equations defined by an ideal $K \subset \mathbb{C}[\partial_1, \dots, \partial_n]$, the dimension of the space of polynomial solutions of K is nontrivial precisely when $\langle \partial_1, \dots, \partial_n \rangle$ is a minimal prime of K (with corresponding primary component K_0) and the dimension of this space equals the dimension of the quotient $\mathbb{C}[\partial_1, \dots, \partial_n]/K_0$ as a complex vector space (cf. [Stu02, Section 10.3]). We deduce that the multiplicity of $I(M)$ at the origin equals the number of bounded M-subgraphs of $\mathbb{N}^{\bar{J}}$.

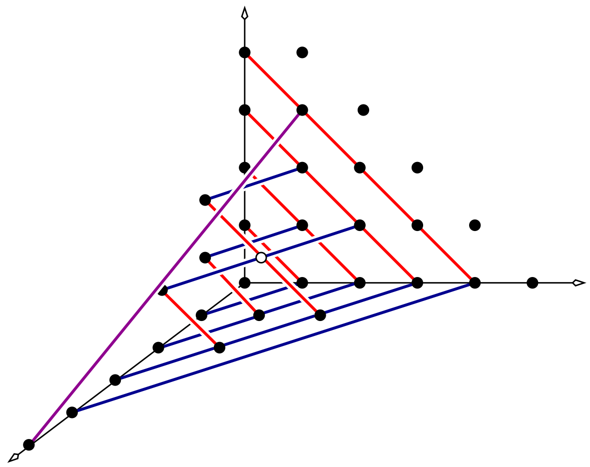

FIGURE 2. The M-subgraphs of $\mathbb{N}^3$

We can use this correspondence between M-subgraphs and solutions of $I(M)$ to compute examples.

Example 7.8. Consider the 3×3 matrix

$$M = \begin{bmatrix} 1 & -5 & 0 \\ -1 & 1 & -1 \\ 0 & 3 & 1 \end{bmatrix}$$

A basis of solutions (with irreducible supports) of $I(M)$ is easily computed:

$$\left\{ 1, \quad x+y+z, \quad (x+y+z)^2, \quad (x+y+z)^3, \quad \sum_{n \geq 4} \frac{(x+y+z)^n}{n!} \right\}.$$

The M-subgraphs of $\mathbb{N}^3$ are the four slices $\{(a,b,c) \in \mathbb{N}^3 : a+b+c=n\}$ for $n \leq 3$; for $n \geq 4$, two consecutive slices are M-connected by $(-5,1,3)$, yielding one unbounded M-subgraph.

Remark 7.9. The system $I(M)$ is itself a binomial Horn system; there are no Euler operators because M is invertible. We stress that it is a very special feature of hypergeometric differential equations that their irreducible (Puiseux) series solutions are determined (up to a constant multiple) by their supports. In general, this is far from being the case for systems of differential equations that are not hypergeometric.

The following definition will allow us to determine a set of parameters β for which the system $H_A(\mathcal{C}_{\rho,J}, \beta)$ has the explicit basis of solutions that we construct for Theorem 7.14.

Definition 7.10. A *facet* of A_J is a subset of its columns that is maximal among those minimizing nonzero linear functionals on $\mathbb{Z}^d$. For a facet σ of A_J let ν_σ be its *primitive support function*, the unique rational linear form satisfying

(1) $\nu_\sigma(\mathbb{Z}A_J) = \mathbb{Z}$,
(2) $\nu_\sigma(a_j) \geq 0$ for all $j \in J$,
(3) $\nu_\sigma(a_j) = 0$ for all $a_j \in \sigma$.

A parameter vector $\beta \in \mathbb{C}^d$ is A_J*-nonresonant* if $\nu_\sigma(\beta) \notin \mathbb{Z}$ for all facets σ of A_J. Note that if β is A_J-nonresonant, then so is $\beta + A_J(\gamma)$ for any $\gamma \in \mathbb{Z}^J$.

The reason nonresonant parameters are convenient to work with is the following.

Lemma 7.11. *If β is A_J-nonresonant, then for any $\gamma \in \mathbb{N}^J$, and for all torus translates I_ρ of the toric ideal I_{A_J}, right multiplication by ∂_J^γ induces a D_J-module isomorphism $D_J/H_{A_J}(I_\rho, \beta) \to D_J/H_{A_J}(I_\rho, \beta + A_J(\gamma))$, whose left inverse we denote by $\partial_J^{-\gamma}$.*

Proof. For $R = \mathbb{C}[\partial_J]/I_\rho$ there is an exact sequence $0 \to R \xrightarrow{\partial_J^\gamma} R \to R/\partial_J^\gamma R \to 0$. Since the multiplication by ∂_J^γ occurs in the right-hand factor of $D_J \otimes_{\mathbb{C}[\partial_J]} R$, the map on Euler-Koszul homology over D_J induced by ∂_J^γ corresponds to right multiplication. But $R/\partial_J^\gamma R$ is toral by Lemma 4.3, and its set of quasidegrees is the Zariski closure of $\{-A_J\vartheta : \vartheta \in \mathbb{N}^J, \vartheta_i < \gamma_i \text{ for some } i \in J\}$, which is a finite subspace arrangement contained in the resonant parameters. Now apply Lemma 2.3 and Theorem 4.5 to complete the proof. □

The following definition characterizes parameter vectors with particularly nice behavior when it comes to isomorphisms between $H_A(I, \beta)$ for varying β.

Definition 7.12. A parameter vector $\beta \in \mathbb{C}^d$ is called *very generic* if $\beta - A_{\bar{J}}(\gamma)$ is A_J-nonresonant for every $\gamma \in \mathcal{S}(M)$.

Remark 7.13. Denote by $\mathrm{Sol}(I_\rho, \beta)$ the space of local holomorphic solutions of $H_{A_J}(I_\rho, \beta)$ near a nonsingular point. Given $\alpha \in \mathbb{N}^J$, the D-module isomorphism in Lemma 7.11 induces a vector space isomorphism

$$\mathrm{Sol}(I_\rho, \beta) \longleftarrow \mathrm{Sol}(I_\rho, \beta + A_J\alpha)$$

given by differentiation by ∂_J^α. If we denote the inverse of this map by $\partial_J^{-\alpha}$, a number of questions arise: for instance, given a local solution $f \in \mathrm{Sol}(I_\rho, \beta)$ where β is very generic, and taking $J = \{1, 2\}$:

(1) Is $\partial_{\{1,2\}}^{-(1,0)}\big(\partial_{\{1,2\}}^{-(0,1)} f\big)$ equal to $\partial_{\{1,2\}}^{-(1,1)} f$?
(2) Is $\partial_{\{1,2\}}^{(1,1)}\big(\partial_{\{1,2\}}^{-(2,2)} f\big)$ equal to $\partial_{\{1,2\}}^{-(1,1)} f$?

Both questions have positive answers; their verification is based on the fact that the left and right inverses of a vector space isomorphism are the same. We conclude that $\partial_J^{-\alpha} f$ is well-defined for any $f \in \mathrm{Sol}(I_\rho, \beta + A_J\alpha)$, if β is very generic and α is an arbitrary integer vector. At the level of D-modules, however, $\partial_J^{-\alpha}$ for $\gamma \in \mathbb{Z}^J$ is not necessarily well-defined, because the right and the left inverses of a D-isomorphism need not coincide.

We resume Notation 7.5. If $q = 0$, then set $G_\varnothing = 1$. If $q > 0$ and $\gamma \in \mathcal{S}(M)$, then rewrite the polynomial G_γ from Lemma 7.4 as follows:

$$G_\gamma = x_{\bar{J}}^\gamma \sum_{\gamma + Mv \in \Gamma} c_v x_{\bar{J}}^{Mv}.$$

By Proposition 7.6, $\{G_\gamma : \gamma \in \mathcal{S}(M)\}$ is a basis for the polynomial solution space of $I(M)$.

Given a local solution $f = f(x_J)$ of $H_{A_J}(I_\rho, \beta - A_{\bar{J}}(\gamma))$ for some $\gamma \in \mathcal{S}(M)$, define

$$F_{\gamma,f} = x_{\bar{J}}^{\gamma} \sum_{\gamma + Mv \in \Gamma} c_v x_{\bar{J}}^{Mv} \partial_J^{-Nv}(f), \tag{7.1}$$

where $\partial_J^{-Nv} f$ is as in Remark 7.13. Note that if $q = 0$, we have $F_{\varnothing,f} = f$.

The condition of being very generic is open and dense in the standard topology of $\mathbb{C}^d$, so that the rank of $H_A(\mathcal{C}_{\rho,J}, \beta)$ for such parameters equals the generic rank of this binomial D-module, in the sense of Theorem 6.10.

Theorem 7.14. *Let $\mathcal{C}_{\rho,J}$ be a toral component of $I(B)$ and let $\beta \in \mathbb{C}^d$ be a very generic parameter vector. Given $\gamma \in \mathcal{S}(M)$, fix a basis $\mathcal{B}_\gamma$ of local solutions of $H_{A_J}(I_\rho, \beta - A_{\bar{J}}(\gamma))$. The $\mu_M \cdot \mathrm{vol}(A_J)$ functions $\{F_{\gamma,f} : \gamma \in \mathcal{S}(M), f \in \mathcal{B}_\gamma\}$ form a local basis for the solution space of the binomial D-module $D/H_A(\mathcal{C}_{\rho,J}, \beta)$.*

Before proving Theorem 7.14, let us see the construction (7.1) in some explicit examples.

Example 7.15. Consider the matrices

$$A = \left[\begin{smallmatrix} 1 & 1 & 1 & 1 & 1 \\ 5 & 10 & 0 & 7 & 6 \end{smallmatrix}\right] \quad \text{and} \quad B^t = \left[\begin{smallmatrix} 0 & -1 & 0 & 4 & -3 \\ -1 & 0 & 1 & 5 & -5 \\ 2 & -1 & -1 & 0 & 0 \end{smallmatrix}\right].$$

We concentrate on the decomposition

$$M = \left[\begin{smallmatrix} 4 & 5 \\ -3 & -5 \end{smallmatrix}\right]; \quad N = \left[\begin{smallmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 1 \end{smallmatrix}\right]; \quad B_J = \left[\begin{smallmatrix} 2 \\ -1 \\ -1 \end{smallmatrix}\right].$$

Note that $\mathbb{Z}B_J$ is saturated, so there is only one associated prime coming from this decomposition, namely $I_{\left[\begin{smallmatrix} 1 & 1 & 1 \\ 5 & 10 & 0 \end{smallmatrix}\right]} + \langle \partial_4, \partial_5 \rangle$, and this is toral since $\det(M) \neq 0$.

The polynomial $\varphi = 5x_4^4x_5^2 + 2x_4^5 + 2x_5^5 + 40x_4x_5^3$ is a solution of the constant coefficient system $I(M)$. Let f be a local solution of the $\left[\begin{smallmatrix} 1 & 1 & 1 \\ 5 & 10 & 0 \end{smallmatrix}\right]$-hypergeometric system that is homogeneous of degree $\beta - \left[\begin{smallmatrix} 6 \\ 40 \end{smallmatrix}\right]$. Then, the following function is a solution of $H(B, \beta)$:

$$5x_4^4x_5^2 f + 2x_4^5 \partial_1^1 \partial_2^{-1} \partial_3^{-1} f + 2x_5^5 \partial_2^{-1} f + 40x_4x_5^3 \partial_1 \partial_2^{-2} \partial_3^{-1} f.$$

In this example, the new solution we constructed has 1-dimensional support.

Example 7.16. Our procedure for constructing solutions works even when M is an $m \times m$ matrix, i.e., M is a maximal square submatrix of B. For instance, consider

$$A = \left[\begin{smallmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 2 \end{smallmatrix}\right] \quad \text{and} \quad B^t = \left[\begin{smallmatrix} 2 & -1 & 0 & -1 \\ -3 & 2 & 1 & 0 \end{smallmatrix}\right].$$

We concentrate on the component

$$M = \left[\begin{smallmatrix} 2 & -3 \\ -1 & 2 \end{smallmatrix}\right]; \quad N = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]; \quad B_J = \varnothing.$$

Again, we only have one (toral) component, associated to $\langle \partial_1, \partial_2 \rangle$. Let $p = x_1^2 + 2x_2$. This is a solution of $I(M) = \langle \partial_1^2 - \partial_2, \partial_1^3 - \partial_2^2 \rangle$. Since B_J is empty, we need only consider solutions of the homogeneity equations that are functions of x_3 and x_4. Since $\det(M) = 1 \neq 0$, the complementary minor of A is also nonzero, and therefore there exists a unique monic monomial in x_3 and x_4 of each degree. To make a solution of $I + \langle E - \beta \rangle$, let $x_3^{w_3}x_4^{w_4}$ be the unique monic solution of the homogeneity equations with parameter $\beta - \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. Then

$$w_4x_1^2x_3^{w_3}x_4^{w_4-1} + 2x_2x_3^{w_3}x_4^{w_4} = x_3^{w_3}x_4^{w_4-1}(w_4x_1^2 + 2x_2)$$

is the desired solution of $H(B, \beta)$.

Proof of Theorem 7.14. First note that if $q = 0$, all of the statements hold by construction. Therefore we assume that $q \geq 2$.

It is clear that all of the binomial generators of I_ρ annihilate $F_{\gamma,f}$. It is also easy to check that $F_{\gamma,f}$ satisfies the desired homogeneity equations. Let then $(\nu, \delta) \in \mathbb{Z}^J \times \mathbb{Z}^{\bar{J}}$ be one of the q columns of B involving N and M; i.e., $(\nu, \delta) = \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] e_k$ for some $k \in \bar{J}$. To prove that $(\partial_J^{\nu_+}\partial_{\bar{J}}^{\delta_+} - \partial_J^{\nu_-}\partial_{\bar{J}}^{\delta_-})(F_{\gamma,f}) = 0$, notice that $(\partial_{\bar{J}}^{\delta_+} - \partial_{\bar{J}}^{\delta_-})(G_\gamma) = 0$. Therefore, for all v with $c_v \neq 0$, either $\partial_{\bar{J}}^{\delta_+}(x^{\gamma+Mv}) = 0$ or there exists another integer vector w with $c_w \neq 0$ such that

$$\partial_{\bar{J}}^{\delta_+}(c_v x_{\bar{J}}^{\gamma+Mv}) = \partial_{\bar{J}}^{\delta_-}(c_w x_{\bar{J}}^{\gamma+Mw}).$$

In the first case, $\partial_J^{\nu_+}\partial_{\bar{J}}^{\delta_+}\left(c_v x_{\bar{J}}^{\gamma+Mv}\partial_J^{N(-v)}(f)\right) = 0$. In the second case, since the monomials on the left and right-hand sides of the above equation must have the same exponent vector, we see that $\delta = M(v-w) = M \cdot e_k$. But M is invertible by assumption, so that $v - w = e_k$. This implies that $\nu = Ne_k = N(v-w)$.

Consequently, $\partial_J^{N(-v)+\nu_+}(f) = \partial_J^{N(-w)+\nu_-}(f)$, and thus

$$\partial_{\bar{J}}^{\delta_+}\partial_J^{\nu_+}\left(c_v x_{\bar{J}}^{\gamma+Mv}\partial_J^{N(-v)}(f)\right) = \partial_{\bar{J}}^{\delta_-}\partial_J^{\nu_-}\left(c_w x_{\bar{J}}^{\gamma+Mw}\partial_J^{N(-w)}(f)\right).$$

Moreover, it is clear that the $F_{\gamma,f}$ are linearly independent.

Now we need to show that these functions span the local solution space of $H_A(\mathcal{C}_{\rho,J}, \beta)$. Let $F = F(x_1, \ldots, x_n)$ be a local solution of $H_A(\mathcal{C}_{\rho,J}, \beta)$. Here we use the explicit description of $\mathcal{C}_{\rho,J}$ from Example 3.7. Since the monomials in U_M annihilate F, we can write

$$F = \sum_{\gamma \in \mathcal{S}(M)} \sum_{\alpha \in \Gamma} x_{\bar{J}}^{\alpha} h_\alpha(x_J),$$

where the sum runs over the union of the bounded M-subgraphs, that is, the sum runs over all α such that $\partial_{\bar{J}}^{\alpha}$ does not belong to U_M. The functions h_α are solutions of $H_A(I_\rho, \beta - A_{\bar{J}}(\alpha))$, as is easy to check. Note that h_α may be zero.

Now it is time to use the equations $I(B)$. First, observe that we may assume that the $x_{\bar{J}}$-monomials in F belong to a single M-subgraph of $\mathbb{N}^{\bar{J}}$. This is because the only equations relating different summands from F are those from $I(B)$, which will relate a summand $x_{\bar{J}}^{\alpha} h_\alpha(x_J)$ to a different summand $x_{\bar{J}}^{\alpha'} h_{\alpha'}(x_J)$ exactly when $\alpha - \alpha'$ or $\alpha' - \alpha$ is a column of M, thus staying within an M-subgraph.

So fix a bounded M-subgraph Γ corresponding to a $\gamma \in \mathcal{S}(M)$, and write

$$F = \sum_{\alpha \in \Gamma} x_{\bar{J}}^{\alpha} h_\alpha(x_J).$$

Fix $\alpha \in \Gamma$ such that $h_\alpha \neq 0$, recall that G_γ is a polynomial solution of $I(M)$ whose support is Γ, and let c be the (nonzero) coefficient of $x_{\bar{J}}^{\alpha}$ in G_γ. We want to show that $F = (1/c)F_{\gamma,h_\alpha}$. Since we know that $F - (1/c)F_{\gamma,h_\alpha}$ has support contained in Γ and has no summand with x^α, the desired equality will be a consequence of the following.

Claim. With the notation above, if $h_\alpha = 0$ then $F = 0$.

Proof of the Claim. If Γ is a singleton, we are done. Otherwise pick $\alpha' \in \Gamma$ such that $\alpha' - \alpha$ or $\alpha - \alpha'$ is a column of M, say $\alpha - \alpha' = Me_k$. The binomial from the corresponding column of B is $\partial_J^{Ne_{k+}} \partial_{\bar{J}}^{Me_{k+}} - \partial_J^{Ne_{k-}} \partial_{\bar{J}}^{Me_{k-}}$. Since this binomial annihilates F, and $\alpha - (Me_k)_+ = \alpha' - (Me_k)_-$, we have

$$\partial_{\bar{J}}^{(Me_k)_+} x_{\bar{J}}^{\alpha} \partial_J^{(Ne_k)_+} h_\alpha = \partial_J^{(Me_k)_-} x_{\bar{J}}^{\alpha'} \partial^{(Ne_k)_-} h_{\alpha'},$$

so that, as $h_\alpha = 0$, $\quad \partial_{\bar{J}}^{(Me_k)_-} x_{\bar{J}}^{\alpha'} \partial_J^{(Ne_k)_-} h_{\alpha'} = 0.$

Now, the first derivative in the previous expression is nonzero, so $\partial_J^{(Ne_k)_-} h_{\alpha'} = 0$. But then $h_{\alpha'} = 0$, since differentiation in any of the x_J variables is an isomorphism (which is why we need our parameter to be very generic).

Propagate the previous argument along Γ to finish the proof of the claim, and with it the proof of the theorem. □

Remark 7.17. When $\mathcal{C}_{\rho,J}$ is Andean (and β is generic), the above procedure produces no nonzero solutions, as expected, since in this case, $D/H_A(\mathcal{C}_{\rho,J}, \beta) = 0$. The reason that the construction breaks in this situation is that there are no nonzero solutions for the "toric" part.

Corollary 7.18. *Fix B as in Convention 1.2. If there exists a parameter β for which the binomial Horn system $H(B,\beta)$ has finite rank, then for generic parameters β, this rank is*

$$\operatorname{rank}(H(B,\beta)) = \sum_{\mathcal{C}_{\rho,J} \text{ toral}} \mu_M \cdot \operatorname{vol}(A_J) = \sum_{B = \left(\begin{smallmatrix} N & B_J \\ M & 0 \end{smallmatrix}\right)} \mu_M \cdot g(B_J) \cdot \operatorname{vol}(A_J),$$

the former sum being over all toral components $\mathcal{C}_{\rho,J}$ of the lattice basis ideal $I(B)$, and the latter sum being over all decompositions of B as in (3.1) with M invertible. Here, $g(B_J)$ is the cardinality of $\operatorname{sat}(\mathbb{Z}B_J)/\mathbb{Z}B_J$, and μ_M is the number of bounded M-subgraphs of $\mathbb{N}^{\bar{J}}$.

Proof. The first equality is a direct consequence of Theorem 7.14 and Theorem 6.8. Comparing with Theorem 6.10 yields the fact that μ_M equals the multiplicity $\mu_{\rho,J}$ of $I_{\rho,J}$ as an associated prime of $I(B)$. For the second equality, the number of components arising from a decomposition (3.1) is $g(B_J)$ [ES96, Corollary 2.5]. □

Remark 7.19. The only sense in which our rank formula for Horn systems is not completely explicit is that it lacks an expression for the number μ_M of bounded M-subgraphs. In the case that $I(M)$ (or $I(B)$) is a complete intersection, Cattani and Dickenstein [CD07] can be applied to provide an explicit recursive formula for $\mu_M = \mu_{\rho,J}$. The general case—even just the toral case—of this computation is an open problem.

Example 7.20. The existence clause for β in Corollary 7.18 is essential: there exist matrices B for which holonomicity of the Horn system $H(B,\beta)$ fails for all parameters β. Let

$$A = \begin{bmatrix} -3 & -1 & 2 & 1 & 0 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B^t = \begin{bmatrix} 1 & -1 & 1 & 0 & 0 \\ 1 & -2 & 0 & 1 & 0 \\ 1 & -3 & 0 & 0 & 1 \end{bmatrix}.$$

Then $\langle \partial_1, \partial_2 \rangle$ is an Andean prime of $I(B)$. The quasidegree set of the corresponding component is $\mathbb{C} \cdot \left[\begin{smallmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \end{smallmatrix}\right] = \mathbb{C}^2$, which means that the Andean arrangement of $I(B)$ equals $\mathbb{C}^2$, and thus $H(B,\beta)$ is non-holonomic for all parameters β.

A sufficient condition to guarantee holonomicity of $H(B, \beta)$ for generic parameters is to require that $I(B)$ be a complete intersection. This is automatic for $m = 2$, so the following result is a direct generalization of [DMS05, Theorem 8.1].

Proposition 7.21. *If $I(B)$ is a complete intersection, then the binomial Horn system $H(B, \beta)$ is holonomic for generic parameters β.*

Proof. If $I(B)$ is a complete intersection, all associated primes have dimension d. In combinatorial terms, we encounter only square matrices M in the primary decomposition of $I(B)$. The component associated to a decomposition (3.1) is Andean exactly when $\det(M) = 0$, and in this case, the corresponding quasidegree set is a proper subset of $\mathbb{C}^d$. We conclude that the Andean arrangement of $I(B)$ is strictly contained in $\mathbb{C}^d$. □

When $I(B)$ is standard $\mathbb{Z}$-graded and has no Andean components, we can obtain a cleaner rank formula à la Bézout, by noting that the sum in Corollary 7.18 equals the degree of $I(B)$. This is a generalization of a result in [Sad02].

Corollary 7.22. *Assume that $I(B)$ is standard $\mathbb{Z}$-graded and has no Andean components. Let $d_1, \ldots, d_m$ be the degrees of the generators of $I(B)$. Then*

$$\operatorname{rank}(H(B, \beta)) = d_1 \cdots d_m \quad \textit{for all } \beta \in \mathbb{C}^d.$$

Proof. Since $I(B)$ has no Andean components, $\mathbb{C}[\partial]/I(B)$ is toral. Moreover, $\mathbb{C}[\partial]/I(B)$ is Cohen-Macaulay, as $I(B)$ is a complete intersection by Lemma 7.2. Theorem 4.9 implies that the holonomic rank of $H(B, \beta)$ is constant. Now apply Corollary 6.13. □

Example 7.23. [Examples 1.8, 1.10, and 1.15 continued] In this running example, the ideal $I(B)$ has two associated primes, the toric ideal I_A and the ideal $\langle \partial_2, \partial_3 \rangle$, both of which are toral. Thus, by Corollary 7.22, the holonomic rank of $H(B, \beta)$ is always $2 \times 2 = 4$.

It is easy to check directly that the generic rank is 4: if the parameter β lies outside the lines spanned by $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\3\end{smallmatrix}\right]$ (that is, if β lies outside the primary arrangement) then the Puiseux monomial $m_\beta = x_1^{\beta_1 - \beta_2/3} x_4^{\beta_2/3}$ is a solution of $H(B, \beta)$ but not of $H_A(\beta)$. Thus, for any local basis $\{\varphi_1, \varphi_2, \varphi_3\}$ of holomorphic solutions of the A-hypergeometric system $H_A(\beta)$ (which has holonomic rank equal to $3 = \operatorname{vol}(A)$ for all β, as $\mathbb{C}[\mathbb{N}A]$ is Cohen–Macaulay) we have that $\{m_\beta, \varphi_1, \varphi_2, \varphi_3\}$ is a basis of local holomorphic solutions of $H(B, \beta)$.

If β lies in the primary arrangement, then the monomial m_β is a solution of $H_A(\beta)$, and the fourth solution of $H(B, \beta)$, like the two logarithmic solutions in Example 6.14, does not come from the associated systems $H_A(\beta)$ and $\langle \partial_2, \partial_3 \rangle + \langle E - \beta \rangle$.

In the standard $\mathbb{Z}$-graded case, binomial D-modules are regular holonomic. The method of canonical series solutions [SST00] then produces expansions for their solutions into power series with logarithms. This method applies to any regular holonomic D-ideal, not just those of the form $H_A(I, \beta)$ for $\mathbb{Z}$-graded I. However, in the binomial D-module case, for very generic parameters the *supports* of the series solutions (i.e., the sets of exponents of the monomials appearing with nonzero coefficients in the series) can be explicitly described, similarly to the combinatorial descriptions for GKZ functions (see [GKZ89] or [SST00]).

Definition 7.24. Let $L \subseteq \mathbb{Z}^n$ be a rank m lattice and $\alpha \in \mathbb{C}^n$. A formal series $x^\alpha \sum_{u \in L} c_u x^u$ is *fully supported* on $\alpha + L$ if there exists an m-dimensional polyhedral cone $C \subseteq \mathbb{R}^n$, a vector $\lambda \in L$, and a sublattice $L' \subseteq L$ of full rank m such that every term $c_u x^{\alpha+u}$ for $u \in (\lambda + C) \cap L'$ has nonzero coefficient c_u.

In our case, the lattice L comes from a toral component $\mathcal{C}_{\rho,J}$ of a $\mathbb{Z}$-graded lattice basis ideal $I(B)$ corresponding to a decomposition (3.1), and sublattices L' are necessary because L is often the saturation of some other given lattice (such as $\mathbb{Z}B$). Recall that $\rho : L \to \mathbb{C}^*$ is a partial character, where the lattice $L \subseteq \mathbb{Z}^J \subseteq \mathbb{Z}^n$ is the saturation of the integer span of the columns of B_J in the notation from (3.1); equivalently, $L = \ker_{\mathbb{Z}}(A_J)$ by Lemma 3.4.

Corollary 7.25. *Fix $\gamma \in \mathcal{S}(M)$ as in Notation 7.5, and let $\Lambda \subseteq \mathbb{N}^n$ be a B-subgraph whose projection to the $\bar{J}$ coordinates is the bounded M-subgraph containing γ. If $\beta \in \mathbb{C}^d$ is very generic, then there exists a vector $\alpha \in \mathbb{C}^J \subseteq \mathbb{C}^n$, unique modulo L, such that $A(\alpha + \Lambda) = \beta$. The* $\mathrm{vol}(A_J)$ *functions $\{F_{\gamma,f} : f \in \mathcal{B}_\gamma\}$ from Theorem 7.14 are fully supported on $\alpha + \Lambda + L$.*

Proof. First note that $A\Lambda$ is a well defined point in $\mathbb{Z}^d$, as two elements of Λ differ by an element of $\mathbb{Z}B \subseteq \ker_{\mathbb{Z}}(A)$. It follows that $\Lambda + L = \lambda + L$ for any $\lambda \in \Lambda$, so it makes sense to be fully supported on $\alpha + \Lambda + L$. On the other hand, the linear system $A\alpha = -A\Lambda + \beta$ has a unique solution modulo (the complex span of) L since A_J has full rank; recall that we are working with a toral component. Now the statement about the supports follows from Theorem 7.14, since elements of $\mathcal{B}_\gamma$ are expressible as series on $\alpha + \Lambda + L$ that are fully supported—either as Gamma series à la [GKZ89] or as canonical series à la [SST00]. □

Remark 7.26. We saw in the Introduction that a solution of $H_A(\beta)$ (or any of the binomial D-modules arising from a torus translate of I_A) is essentially a function in $m = n - d$ variables. Therefore, for generic β, the supports of canonical series expansions as in [SST00], are translates cones of dimension m. This implies that the support of the series (7.1) has dimension $m - q$, since the dimension of the support equals that of any series expansion of f. In fact, this support might not be the set of lattice points in a cone, but in a polyhedron whose recession cone has the correct dimension. Nonetheless, the only fully supported solutions of $H_A(I(B), \beta) = H(B, \beta)$ arise from $H_A(I_{\mathbb{Z}B}, \beta)$. Interestingly, there can be no solutions with support of dimension $m - 1$, because a matrix with $q = 1$ row is never mixed.

Remark 7.27. The ideas above can be used to provide an analogous combinatorial description for the supports of certain solutions of $H_A(I, \beta)$ when I is a general $\mathbb{Z}$-graded binomial ideal. The key observation is that if $\mathcal{C}_{\rho,J}$ is a toral primary ideal, and the parameter $\beta \in \mathbb{C}^d$ is very generic inside $\mathrm{qdeg}(\mathbb{C}[\partial]/\mathcal{C}_{\rho,J})$, then the solutions of $H_A(\mathcal{C}_{\rho,J}, \beta)$ are supported on translates of the L-bounded components, where $L \subseteq \mathbb{Z}^J$ is the underlying lattice of ρ. When $\mathcal{C}_{\rho,J}$ is a primary component of I, this allows us to assert a lower bound on the number of series solutions of $H_A(I, \beta)$ with the desired support. Care must be taken because $\mathrm{qdeg}(\mathbb{C}[\partial]/\mathcal{C}_{\rho,J})$ could be partially or entirely contained in the jump arrangement (Definition 6.9), or $I_{\rho,J}$ could be an embedded prime, in which case the rank at β need not equal a sum of multiplicities times volumes.

REFERENCES

[Ado94] Alan Adolphson, *Hypergeometric functions and rings generated by monomials*, Duke Math. J. **73** (1994), no. 2, 269–290.

[Ado99] Alan Adolphson, *Higher solutions of hypergeometric systems and Dwork cohomology*, Rend. Sem. Mat. Univ. Padova **101** (1999), 179–190.

[BvS95] Victor V. Batyrev and Duco van Straten, *Generalized Hypergeometric Functions and Rational Curves on Calabi-Yau Complete Intersections in Toric Varieties*, Commun. Math. Phys. **168** (1995). 493–533.

[Bjö79] Jan Erik Björk, *Rings of differential operators*, North-Holland Mathematical Library, vol. 21, North-Holland Publishing Co., Amsterdam, 1979.

[BH93] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993.

[CD07] Eduardo Cattani and Alicia Dickenstein, *Counting solutions to binomial complete intersections*, J. of Complexity, Vol. **23**, Issue 1, Feb. 2007, 82–107.

[DMM08] Alicia Dickenstein, Laura Felicia Matusevich, and Ezra Miller, *Combinatorics of binomial primary decomposition*, math.AC/08033846, to appear: Math. Z.

[DMS05] Alicia Dickenstein, Laura Felicia Matusevich, and Timur Sadykov, *Bivariate hypergeometric D-modules*, Adv. Math. **196** (2005), no. 1, 78–123.

[DS07] Alicia Dickenstein and Timur Sadykov, *Bases in the solution space of the Mellin system*, Mat. Sbornik **198:9** (2007), 59-80.

[Erd50] Arthur Erdélyi, *Hypergeometric functions of two variables*, Acta Math. **83** (1950), 131–164.

[ES96] David Eisenbud and Bernd Sturmfels, *Binomial ideals*, Duke Math. J. **84** (1996), no. 1, 1–45.

[FS96] Klaus G. Fischer and Jay Shapiro, *Mixed matrices and binomial ideals*, J. Pure Appl. Algebra **113** (1996), no. 1, 39–54.

[GGZ87] I. M. Gel′fand, M. I. Graev, and A. V. Zelevinskiĭ, *Holonomic systems of equations and series of hypergeometric type*, Dokl. Akad. Nauk SSSR **295** (1987), no. 1, 14–19.

[GKZ89] I. M. Gel′fand, A. V. Zelevinskiĭ, and M. M. Kapranov, *Hypergeometric functions and toric varieties*, Funktsional. Anal. i Prilozhen. **23** (1989), no. 2, 12–26. Correction in ibid, **27** (1993), no. 4, 91.

[GM92] John P. C. Greenlees and J. Peter May, *Derived functors of I-adic completion and local homology*, J. Algebra **149** (1992), no. 2, 438–453.

[Ho99] R. P. Horja, *Hypergeometric functions and mirror symmetry in toric varieties*, math.AG/9912109.

[Hos06] Shinobu Hosono, *Central charges, symplectic forms, and hypergeometric series in local mirror symmetry*, in: Mirror Symmetry V, AMS/IP Studies in Advanced Mathematics, Vol. 38, 2006, 405–440.

[HLY96] S. Hosono, B. H. Lian, and S. T. Yau, *GKZ-Generalized Hypergeometric Systems in Mirror Symmetry of Calabi-Yau Hypersurfaces*, Comm. Math. Phys. **182** (1996), no. 3, 535–577.

[Hot91] Ryoshi Hotta, *Equivariant D-modules*, math.RT/980502.

[HS00] Serkan Hoşten and Jay Shapiro, *Primary decomposition of lattice basis ideals*, J. Symbolic Comput. **29** (2000), no. 4-5, 625–639, Symbolic computation in algebra, analysis, and geometry (Berkeley, CA, 1998).

[Mel21] Hjalmar Mellin, *Résolution de l'équation algébrique générale à l'aide de la fonction* Γ, C.R. Acad. Sc. **172** (1921), 658–661.

[Mil02b] Ezra Miller, *Graded Greenlees–May duality and the Čech hull*, Local cohomology and its applications (Guanajuato, 1999), Lecture Notes in Pure and Appl. Math., vol. 226, Dekker, New York, 2002, pp. 233–253.

[MMW05] Laura Felicia Matusevich, Ezra Miller, and Uli Walther, *Homological methods for hypergeometric families*, J. Amer. Math. Soc. **18** (2005), no. 4, 919–941.

[MS05] Ezra Miller and Bernd Sturmfels, *Combinatorial commutative algebra*, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2005.

[Oku06] Go Okuyama, *A-Hypergeometric ranks for toric threefolds*, Internat. Math. Res. Notices (2006), Article ID 70814, 1–38.

[Sad02] Timur Sadykov, *On the Horn system of partial differential equations and series of hypergeometric type*, Math. Scand. **91** (2002), no. 1, 127–149.

[SST00] Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama, *Gröbner Deformations of Hypergeometric Differential Equations*, Springer-Verlag, Berlin, 2000.

[SW08] Mathias Schulze and Uli Walther, *Irregularity of hypergeometric systems via slopes along coordinate subspaces*, Duke Math. J. **142**,3 (2008), 465–509

[Stu00] Bernd Sturmfels, *Solving algebraic equations in terms of $\mathcal{A}$-hypergeometric series*, Discrete Math. **210** (2000), no. 1-3, 171–181, Formal power series and algebraic combinatorics (Minneapolis, MN, 1996).

[Stu02] Bernd Sturmfels, *Solving Systems of Polynomial Equations*, CBMS Regional Conference Series in Mathematics, Number 97, American Mathematical Society, 2002.

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