

METRIC COMBINATORICS OF CONVEX POLYHEDRA: CUT LOCI AND NONOVERLAPPING UNFOLDINGS

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ABSTRACT. Let S be the boundary of a convex polytope of dimension $d + 1$, or more generally let S be a *convex polyhedral pseudomanifold*. We prove that S has a polyhedral nonoverlapping unfolding into \mathbb{R}^d , so the metric space S is obtained from a closed (usually nonconvex) polyhedral ball in \mathbb{R}^d by identifying pairs of boundary faces isometrically. Our existence proof exploits geodesic flow away from a source point $v \in S$, which is the exponential map to S from the tangent space at v . We characterize the *cut locus* (the closure of the set of points in S with more than one shortest path to v) as a polyhedral complex in terms of Voronoi diagrams on facets. Analyzing infinitesimal expansion of the wavefront consisting of points at constant distance from v on S produces an algorithmic method for constructing Voronoi diagrams in each facet, and hence the unfolding of S . The algorithm, for which we provide pseudocode, solves the discrete geodesic problem. Its main construction generalizes the source unfolding for boundaries of 3-polytopes into \mathbb{R}^2 . We present conjectures concerning the number of shortest paths on the boundaries of convex polyhedra, and concerning continuous unfolding of convex polyhedra. We also comment on the intrinsic non-polynomial complexity of nonconvex manifolds.

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